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What is the function?

MICHAEL DE VILLIERS

University of KwaZulu-Natal, South Africa

e-mail: profmd@mweb.co.za

http://dynamicmathematicslearning.com/homepage4.html

Calculus as the study of the variation and behaviour of functions is an immensely powerful tool with which we can model and understand many real world phenomena. In fact, historically its origins in the 1600's and onwards were strongly rooted in the scientific desire to understand problems of speed, time, acceleration and forces. Unfortunately, calculus is often nowadays taught somewhat divorced from these historical roots. Such real world applications, however, could be a strong source for motivating students and gaining their interest.

Here is one example from an introductory course on differential equations that I recently created using the software *Measurement in Motion*. This software comes with several sample video clips (but one can also import one's own) and some tools with which one can do measurements on the video clip, and then plot these against time, do calculations upon, etc.

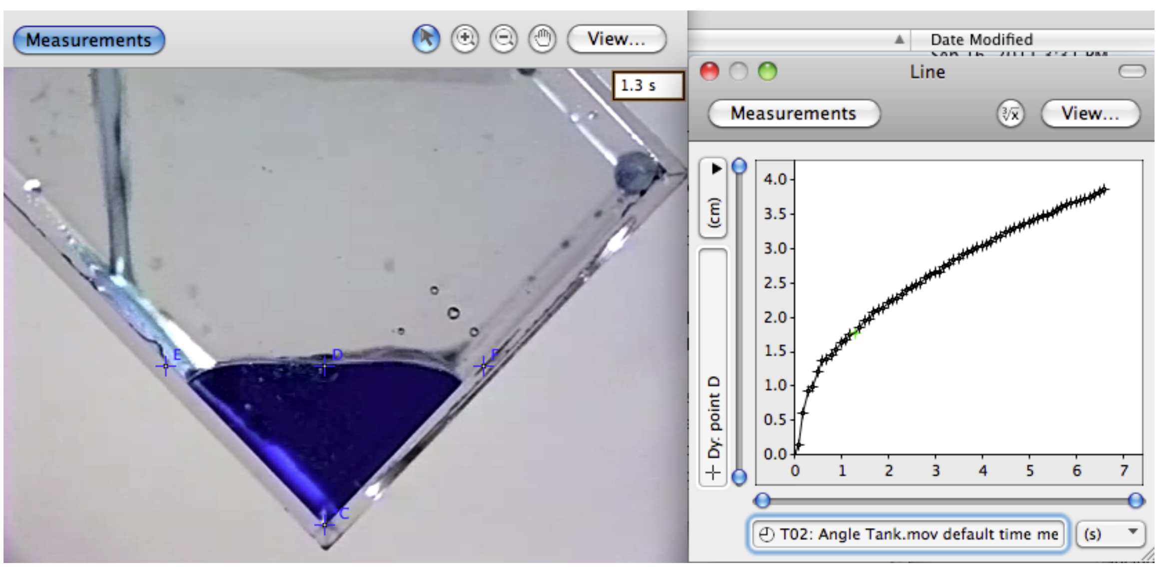

Figure 1

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Figure 1 shows a snapshot of the video clip as water is being poured at a constant rate into a tilted rectangular container. The height of the water is measured and plotted against time in the graph on the right. The video clip is available on YouTube for online viewing at http://www.youtube.com/watch?v=Up8I924idFI or to download at http://frink.machighway.com/~dynamicm/tilted-water-tank.mp4 Readers are more than welcome to use it in their own teaching.

It should be pointed that I prefer first showing students just the video clip without the graph being drawn simultaneously[1], and asking them to describe in their own words how the height varies with time as water is being poured in at a constant rate, and encouraging them to drawing a rough graph of what they think it would look like. This allows them to make a conjecture about the relationship and though some may quickly notice that the height changes more quickly at the start than later, many seem to find it quite challenging to describe the relationship and draw a possible rough graph. For example, the following were the results from a class of 31 students:

* 58% wrote something like "*as time increases the height increases*", "*there is a direct proportional relationship between height and time*", etc. and just drew a linear increasing function from the origin
* only 29% drew more or less correct concave increasing graphs showing a gradually decreasing slope, though their written descriptions varied from the well formulated "*the container first fills up quickly and then takes longer to fill up*" to the poorly formulated "*the height decreases as the glass increases in size, so in a nutshell the height decreased with time*". (The latter student probably meant the 'rate of change' of the height decreases as evidenced by the correct drawing of the graph, but expressed him/herself inaccurately).
* one student noticed the decrease in the 'rate of change' of height writing "*the change of height of water … it decreases as water and time increases*", and then mistakenly drew a linear decreasing graph (starting from some positive value to zero), which might represent dh/dt against time, but clearly not the height against time

[1] Readers can also view the original, unedited video clip online at YouTube at: http://www.youtube.com/watch?v=WBk5-jWUxiw or download or view directly from: http://frink.machighway.com/~dynamicm/angle-tank.mov (*QuickTime* is needed for viewing, but runs on all operating systems & can be freely downloaded)

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* one student drew a convex increasing function and wrote "*as the time goes the liquid rises*", apparently not observing that the rate of change of the height is not increasing, but decreasing with time
* one student drew a graph which roughly appears concave and increasing, though wrote down the following contradictory description "*the height of the water increases faster than the time*"
* one student drew a linear increasing function with it becoming horizontal at the top, but with the student writing '*when water come to an end … the height will decrease*" presumably meaning to say that the height will remain constant

As can be seen from the above excerpts, several students had great difficulty writing down in words what they observed as the relationship between height and time; one even ignoring it completely, and just writing down "*I see a blue liquid in the glass, this liquid form a triangle that have three equal sides*" (yet drawing a linear increasing graph). This shows that many of these 3rd (and some 4th) year mathematics education students, despite knowing and being quite proficient in the 'rules' and 'techniques' of differentiating and integrating, have little conceptual understanding of graphs and the slope or rate of change of a function, especially as it relates to a real world context.

For the majority of students, the plotted graph of the height against time therefore came as an unexpected surprise, as it is not linear as they had thought. This surprise immediately raised the interesting question of WHY that is so, and students seemed really eager for an explanation. More-over, they were now much more intrigued and motivated than usual to find out what type of function is the height as a function of time.

One of the first steps often in modeling is to make a sketch to represent the situation as shown in Figure 2.

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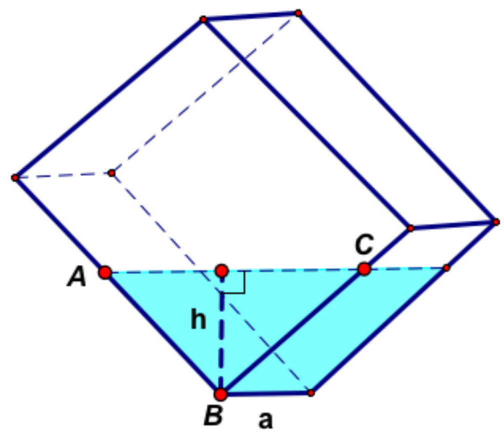

Figure 2

Clearly the volume of water is given by the area of the triangular face *ABC* multiplied by the side length *a*. Conveniently assuming that the rectangular container is tilted at an angle of 45°, it then follows that the volume is given by $V = h^2 a$ where *h* is the height of the right triangular face *ABC* as indicated. Differentiating with respect to *t*, (implicitly on the right hand side) gives: $\frac{dV}{dt} = 2ha\frac{dh}{dt}$. Setting $\frac{dV}{dt} = k$ since it is a constant, gives us the differential equation $k = 2ha\frac{dh}{dt}$, from which we need to solve *h* as a function of *t*. This is a standard exercise in separating variables and integrating, for example:

$$\int h\,dh = \int \frac{k}{2a}dt$$

$$\Rightarrow \frac{1}{2}h^2 = \frac{k}{2a}t + C$$

$$\Rightarrow h = \sqrt{\frac{k}{a}t + 2C}$$

Since at $t = 0$, $h = 0$, it follows that $C = 0$. So we find that that the height *h* is simply a square root function of time, and therefore explains the shape of the experimentally plotted graph. An example of a square root function drawn with *Sketchpad* is given in Figure 3 with the assumption that $k/a = 2$.

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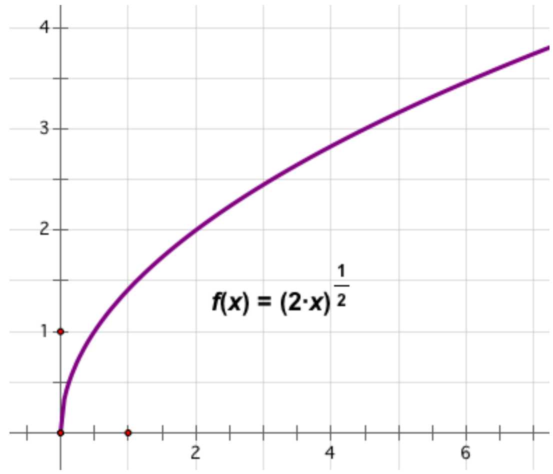

Figure 3

One can now explore similar situations if water is poured at a constant rate into a cylindrical, conic or pyramidal container, for example. With a video camera or even a cell phone nowadays, one could record clips of these and import them into *Measurement in Motion* to investigate each situation experimentally, either before as in this case, or afterwards to corroborate the theory. Using such video clips may certainly help to link mathematics with the real world in a more meaningful way than just standard talk and chalk, and make it "alive".

Reference

Measurement in Motion at http://www.motion.com/products/measurement/index.html