Invited plenary presented at the 4th *Congress of teachers of mathematics* of the Croatian Mathematical Society, Zagreb, 30 June –2 July 2010, as well as at the *National Mathematics Congress* in Namibia, Swakopmund, 21-23 May 2012. Also presented as public lecture in the Program Post-Graduate Studies in Mathematics Education, Pontifical Catholic University of São Paulo, Brazil. 19 April 2010.

Some Reflections on the Van Hiele theory

Michael de Villiers
Mathematics Education, University of KwaZulu-Natal, South Africa

This paper gives a review of research on the Van Hiele Theory over the past 30 years, and highlights some important issues regarding theoretical implications for designing learning activities in dynamic geometry contexts, as well as issues of further research such as hierarchical class inclusion.

The Van Hiele theory originated in the respective doctoral dissertations of Dina van Hiele-Geldof and her husband Pierre van Hiele at the University of Utrecht, Netherlands in 1957. Dina unfortunately died shortly after the completion of her dissertation, and Pierre was the one who developed and disseminated the theory further in later publications.

While Pierre's dissertation mainly tried to explain why pupils experienced problems in geometry education (in this respect it was **explanatory** and **descriptive**), Dina's dissertation was about a teaching experiment and in that sense is more **prescriptive** regarding the ordering of geometry content and learning activities of pupils. The most obvious characteristic of the theory is the distinction of five discrete thought levels in respect to the development of pupils' understanding of geometry. Four important characteristics of the theory are summarized as follows by Usiskin (1982:4):

- **fixed order** - The order in which pupils progress through the thought levels is invariant. In other words, a pupil cannot be at level *n* without having passed through level *n-1*.
- **adjacency** - At each level of thought that which was intrinsic in the preceding level becomes extrinsic in the current level.
- **distinction** - Each level has its own linguistic symbols and own network of relationships connecting those symbols.
- **separation** - Two persons who reason at different levels cannot understand each other.

The main reason for the failure of the traditional geometry curriculum was attributed by the Van Hieles to the fact that the curriculum was presented at a higher level than those of the pupils; in other words they could not understand the teacher nor could the teacher understand why they could not understand! Although the Van Hiele theory distinguishes between five different levels of thought, we shall here only focus on the first four levels as they are the most pertinent ones for secondary school geometry. The general characteristics of each level can be described as follows:

Level 1: Recognition

Pupils visually recognize figures by their global appearance. They recognize triangles, squares, parallelograms, and so forth by their shape, but they do not explicitly identify the properties of these figures.

Level 2: Analysis

Pupils start analyzing the properties of figures and learn the appropriate technical terminology for describing them, but they do not interrelate figures or properties of figures.

Level 3: Ordering

Pupils logically order the properties of figures by short chains of deductions and understand the interrelationships between figures (e.g. class inclusions).

Level 4: Deduction

Pupils start developing longer sequences of statements and begin to understand the significance of deduction, the role of axioms, theorems and proof.

Note that in a certain sense the transition from Level 1 to Level 2 involves a transition from an inactive-iconic handling of concepts to a more symbolic one, to use Bruner's familiar concepts. More simply put, the attainment of Level 2 involves the acquisition of the technical language by which the properties of the concept can be described. However, the transition from Level 1 to Level 2 involves more than just the acquisition of language. It involves recognizing certain new relationships between concepts and the refinement and renewal of existing concepts.

For a student to progress from Level 1 to Level 2 regarding a particular topic (e.g. the quadrilaterals), a significant re-arrangement of relationships and a refinement of concepts have to occur. There is therefore far more in this transition than merely a verbalization of intuitive knowledge; the verbalization goes together with a restructuring of knowledge. This restructuring must first occur before students can start exploring the logical relationships between these properties at Level 3. Van Hiele (1973:94) puts it as follows:

"*The network of relations on Level 3 can only be meaningfully established, when the network of relations at Level 2 are adequately established. When the second network of relations are present in such an adequate form, that its structure becomes apparent and one can talk about it with others, then the building blocks for Level 3 are ready*" (Freely translated from Dutch)

Level 3 also represents a completely different network of relations than Level 2. Where the network of relations at the Level 2 involves the *association of properties* with types of figures and relationships between figures according to these properties, the network of relations at the Level 3 involve the *logical relationships* between the properties of figures. The network of relations at the Level 3 no longer refer to concrete, specific figures, nor do they form a frame of reference in which it is asked whether a given figure has certain properties. The typical questions that are asked at Level 3 are whether a certain property follows from another, or can be deduced from a particular subset of properties (in other words whether it could be taken as a definition or is a theorem) or whether two definitions are equivalent.

The network of relations for the First and Second thought levels are therefore quite different (Van Hiele, 1973:90):

"*The reasoning of a logical system belongs to the Third Level of thought. The network of relations, which is based on a verbal description of observed facts, belong to the Second Level of thought. These two levels have their own networks of relations where the one is distinct from the other: one either reasons in the one network of relations or in the other*."

The differences between the first three levels can be summarized as shown in Table 1 in terms of the objects and structure of thought at each level (adapted from Fuys et al, 1988:6).

Objects of thought	**Individual figures**	**Classes of figures**	**Definitions of classes of figures**
Structure of thought	Visual recognition Naming Visual sorting	Recognizing properties as characteristics of classes	Noticing & formulating logical relationships between properties
Examples	• Parallelograms all go together because they "*look the same*" • Rectangles, squares and rhombi are not parallelograms because they do "*not look like them*"	A parallelogram has: • 4 sides • opp. angles = • opp. sides = • opp. sides // • bisecting diagonals; etc. A rectangle is not a parallelogram since a rectangle has 90° angles, but a parallelogram not.	• Opposite sides = imply opposite sides // • Opposite sides // imply opposite sides = • opposite angles = imply opp. sides = • bisecting diagonals imply half-turn symmetry

Table 1

By using task-based interviews, Burger & Shaughnessy (1986) characterized pupils' thought levels at the first four levels more fully as follows:

Level 1

(1) Often use irrelevant visual properties to identify figures, to compare, to classify and to describe.

(2) Usually refer to visual prototypes of figures, and is easily misled by the orientation of figures.

(3) An inability to think of an infinite variation of a particular type of figure (e,g. in terms of orientation and shape).

(4) Inconsistent classifications of figures; for example, using non-common or irrelevant properties to sort figures.

(5) Incomplete descriptions (definitions) of figures by viewing necessary (often visual) conditions as sufficient conditions.

Level 2

(1) An explicit comparison of figures in terms of their underlying properties.

(2) Avoidance of class inclusions between different classes of figures, eg. squares and rectangles are considered to be disjoint.

(3) Sorting of figures only in terms of one property, for example, properties of sides, while other properties like symmetries, angles and diagonals are ignored.

(4) Exhibit an uneconomical use of the properties of figures to describe (define) them, instead of just using sufficient properties.

(5) An explicit rejection of definitions supplied by other people, e.g. a teacher or textbook, in favour of their own personal definitions.

(6) An empirical approach to the establishment of the truth of a statement; e.g. the use of observation and measurement on the basis of several sketches.

Level 3

(1) The formulation of economically, correct definitions for figures.

(2) An ability to transform incomplete definitions into complete definitions and a more spontaneous acceptance and use of definitions for new concepts.

(3) The acceptance of different equivalent definitions for the same concept.

(4) The hierarchical classification of figures, e.g. quadrilaterals.

(5) The explicit use of the logical form "*if … then*' in the formulation and handling of conjectures, as well as the implicit use of logical rules such as *modus ponens*.

(6) An uncertainty and lack of clarity regarding the respective functions of axioms, definitions and proof.

Level 4

(1) An understanding of the respective functions (roles) of axioms, definitions and proof.

(2) Spontaneous conjecturing and self-initiated efforts to deductively verify them.

Russian research on geometry education

Geometry has always formed an extremely prominent part of the Russian mathematics curriculum in the nineteenth and twentieth centuries. This proud tradition was no doubt influenced by (and instrumental in) the achievements of several famous Russian geometers (like Lobachevsky) in the past two centuries. Traditionally the Russian geometry curriculum consisted of two phases, namely, an *intuitive* phase for Grades 1 to 5 and a *systematization* (deductive) phase from Grade 6 (12/13 year old).

In the late sixties Russian (Soviet) researchers undertook a comprehensive analysis of both the intuitive and the systematization phases in order to try and find an answer to the disturbing question of why pupils who were making good progress in other school subjects, showed little progress in geometry. In their analysis, the Van Hiele theory played a major part. For example, it was found that that at the end of Grade 5 (before the resumption of the systematization phase which requires at least Level 3 understanding) only 10-15% of the pupils were at Level 2.

The main reason for this was the insufficient attention to geometry in the primary school. For example, in the first five years, pupils were expected to become acquainted, via mainly Level 1 activities, with only about 12-15 geometrical objects (and associated terminology). In contrast, it was expected of pupils in the very first topic treated in the first month of Grade 6 to become acquainted not only with about 100 new objects and terminology, but it was also being dealt with at Level 3 understanding. (Or frequently, the teacher had to try and introduce new content at 3 different levels simultaneously). No wonder they described the period between Grades 1 and 5 as a "*prolonged period of geometric inactivity*"!

The Russians subsequently designed a very successful experimental geometry curriculum based on the Van Hiele theory. They found that an important factor was the continuous sequencing and development of concepts from Grade 1. As reported in Wirszup (1976: 75-96), the average pupil in Grade 8 of the experimental curriculum showed the same or better geometric understanding than their Grade 11 and 12 counterparts in the old curriculum.

The primary & middle school geometry curriculum

The parallels from the Russian experience to South Africa and other countries are obvious. In South Africa we used to have a geometry curriculum heavily loaded in the senior secondary school with formal geometry, and with relatively little content done informally in the primary school. (E.g. little similarity or circle geometry is done in the primary school). On average, pupils' performance in the South African matric (Grade 12) geometry was far worse than in algebra. Why?

The Van Hiele theory supplies an important explanation. For example, research by De Villiers & Njisane (1987) showed that about 45% of pupils investigated in Grade 12 in KwaZulu had only mastered Level 2 or lower, whereas the examination assumed mastery at Level 3 and beyond! Similar low Van Hiele levels among secondary school pupils have been found by Malan (1986), Smith & De Villiers (1990) and Govender (1995). In particular, the transition from Level 1 to Level 2 posed specific problems to second language learners, since it involves the acquisition of the technical terminology by which the properties of figures need to be described and explored. This requires sufficient time, which is not available in the presently overloaded secondary curriculum.

It seems clear that no amount of effort and fancy teaching methods at the secondary school will be successful, unless we embark on a major revision of the primary school geometry curriculum along Van Hiele lines. The future of secondary school geometry thus ultimately depends on primary school geometry!

In Japan for example pupils already start off in Grade 1 with extended tangram, as well as other planar and spatial, investigations (e.g. see Nohda, 1992). This is followed up continuously in following years so that by Grade 5 they are already dealing formally with the concepts of congruence and similarity; concepts that are only introduced in Grades 8 and 9 in South Africa. Similarly in Taiwan where geometry is started early, it is reported in a study by Wu & Ma (2006) that 28.3% of Grade 6 learners were already at Van Hiele 3, whereas the same percentage of learners at Van Hiele Level 3 in South Africa, only occurred in Grade 11 (De Villiers & Njisane, 1987; De Villiers, 1987). More recently, Feza & Webb (2005) found that only 5 out of 30 (16.7%) Grade 7 learners interviewed in South Africa, had reached Van Hiele level 2. It seems no wonder that in international comparative studies in recent years, Japanese and Taiwanese school children have consistently outperformed school children from South Africa, as well as other countries.

Although the recent introduction of tessellations in South African primary schools is to be greatly welcomed, many teachers and textbook authors do not appear to understand its relevance in relation to the Van Hiele theory. Although tessellations have great aesthetic attraction due to their intriguing and artistically pleasing patterns, the fundamental reason for introducing it in the primary school is that it provides an intuitive visual foundation (Van Hiele 1) for a variety of geometric content, which can later be treated more formally in a deductive context.

For example, in a triangular tessellation pattern such as shown in Figure 1, one could ask pupils the following questions:

(1) identify and colour in parallel lines

(2) what can you say about angles *A*, *B*, *C* , *D* and *E* and why?

(3) what can you say about angles *A*, *1*, *2*, *3* and *4* and why?

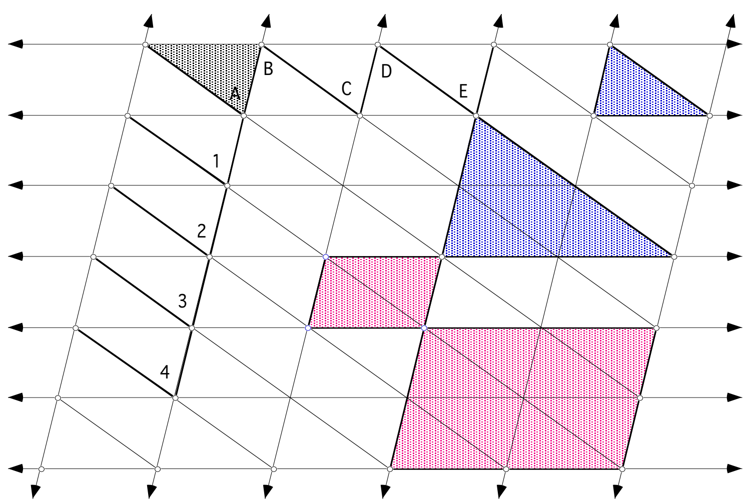


Figure 1: Visualization

In such an activity pupils will realize that angles *A*, *B*, *C*, *D* and *E* are all equal since a halfturn of the grey triangle around the midpoint of the side *AB* maps angle *A* onto angle *B*, etc. In this way, pupils can be introduced for the first time to the concept of "*saws*" or "*zig-zags*" (alternate angles). Similarly, pupils should realize that angles *A*, *1*, *2*, *3* and *4* are all equal since a translation of the grey triangle in the direction of angles *1*, *2*, *3* and *4* consecutively maps angle *A* onto each of these angles. In this way, pupils can be introduced for the first time to the concept of "*ladders*" (corresponding angles). Pupils should further be encouraged to find different saws and ladders in the same and other tessellation patterns to improve their visualization ability.

Since each tile has to be identical and can be made to fit onto each other exactly by means of translations, rotations or reflections pupils can easily be introduced to the concept of congruency. Pupils can also be asked to look for different shapes in such tessellation patterns, e.g. parallelograms, trapezia and hexagons. They could also be encouraged to look for larger figures with the *same shape*, thus intuitively introducing them to the concept of *similarity* (as shown in Figure 1 by the shaded similar triangles and parallelograms).

Tessellations also provide a suitable context for the analysis of the properties of geometric figures (Van Hiele 2), as well as their logical explanation (Van Hiele 3). For example, after pupils have constructed a triangular tessellation pattern as shown in Figure 2, one could ask them questions like the following:

(1) What can you say about angles *A* and *B* in relation to *D* and *E*? Why? What can you therefore conclude from this?

(2) What can you say about angles *F* and *G* in relation to angles *H* and *I*? Why? What can you therefore conclude from this?

(3) What can you say about line segment *JK* in relation to line segment *LM*? Why? What can you therefore conclude from this?

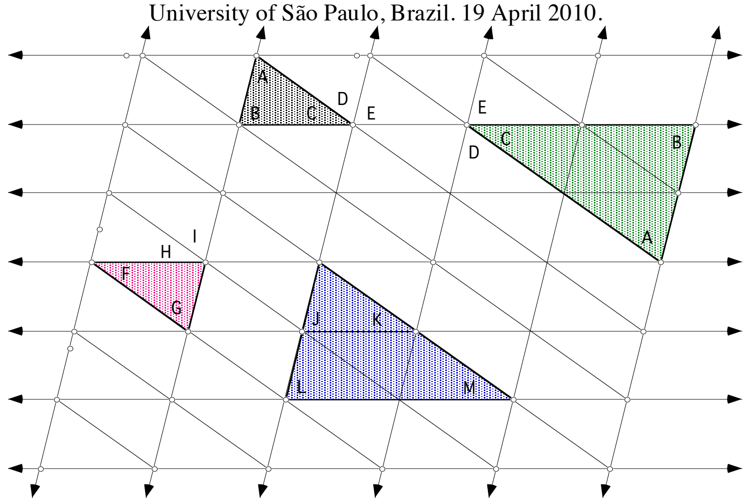


Figure 2: Analyzing

In the first case, pupils can again see that angle A = angle D due to a saw being formed. Also angle B = angle E due to a ladder. It is then easy for them to observe that since the three angles lie on a straight line, that the sum of the angles of triangle ABC must be equal to a straight line. They can also observe that this is true at any vertex, as well as for any size triangle or orientation, thus enabling generalization. In the second case, the exterior angle theorem is introduced and in the third case, the midpoint theorem. Such analyses are clearly just a short step away from the standard geometric explanations (proofs); all they now need is some formalization. In Figure 3 the three levels are illustrated for the discovery and explanation that the opposite angles of a parallelogram are equal.

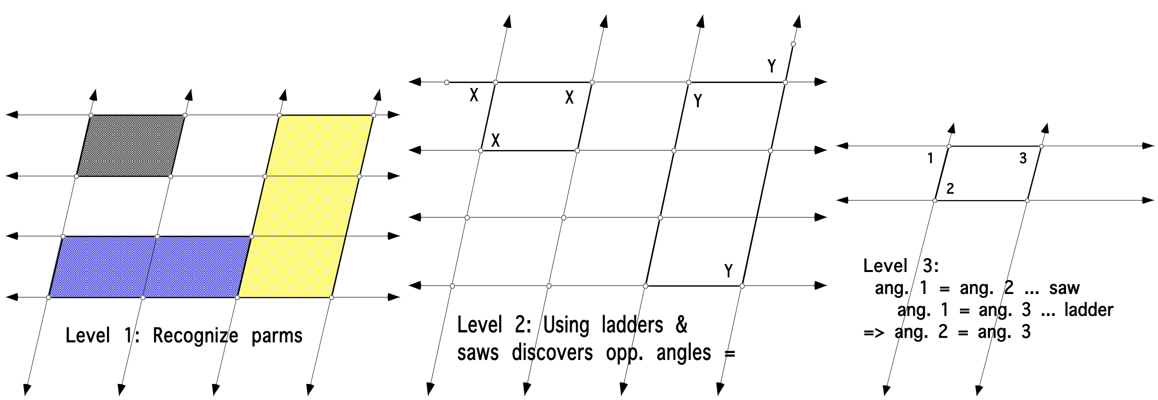


Figure 3: Three levels

Conceptual Structuring

A very important aspect of the Van Hiele theory is that it emphasizes that informal activities at Levels 1 and 2 should provide appropriate "*conceptual substructures*" for the formal activities at the next level. Though different, this idea is somewhat similar to the idea of instructional *scaffolding* promoted by Wood, Bruner & Ross (1976).

Teachers often let their students measure the angles of a triangle with a protractor, and then let them add the angles (usually disregarding 'deviations' as due to experimental error) to 'discover' that they always add up to 180°.

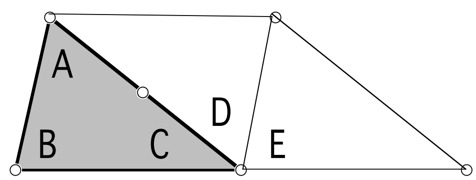


Figure 4: Using transformations to discover

However, from a Van Hiele perspective this is entirely inappropriate as it does not provide a suitable conceptual substructure in which the eventual logical explanation (proof) is implicitly embedded. In comparison, an activity with cardboard tiles or *Sketchpad* like the following from De Villiers (2003) provides such a substructure. For example, translate a triangle *ABC* by vector *BC*, and rotate triangle ABC around the midpoint of *AC* (see Figure 4). Let the students notice through dragging that the three angles at *C*, *D*, and *E* always form a straight line. Then ask students what they can say about angles *A* and *B* in relation to angles *D* and *E* in terms of the transformations carried out. Since angle *B* maps on to angle *E* by the translation, and angle *A* maps to angle *D* by the half-turn, angles *B* and *A* are equal to angles *D* and *E*, respectively. Clearly this provides a much more appropriate conceptual structure for an eventual explanation (proof) than simply letting students measure some angles of triangles.

Similarly, the activity of measuring the base angles of an isosceles triangle is conceptually inappropriate, but folding it around its axis of symmetry lays the foundation for a formal proof later. The same applies to the investigation of the properties of the quadrilaterals. For example, it is conceptually inappropriate to measure the opposite angles of a parallelogram to let pupils discover that they are equal. It is far better to let them give the parallelogram a half-turn to find that opposite angles (and sides) map onto each other, as this generally applies to all parallelograms and contains the conceptual seeds for a formal proof.

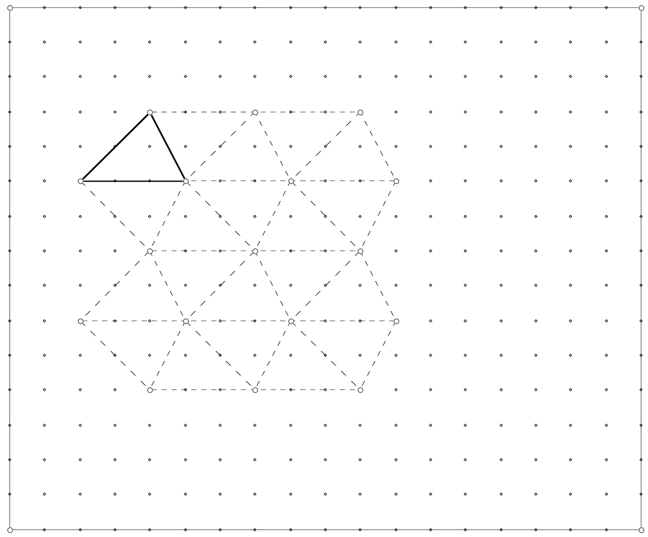


Figure 5: Using grids to produce tessellations

Recently I had a conversation with a teacher who quickly dismissed a fellow teacher's introduction to tessellations who first let his pupils pack out little cardboard tiles. This teacher felt that it produced untidy patterns, was ineffective and time consuming, and that one should just start by providing pupils with ready-made square or triangular grids and show them how they can then easily draw neat tessellation patterns (see Figure 5). Although such grids are a useful and effective way of drawing neat patterns, it is conceptually extremely important for pupils to at least have some experience of physically packing out tiles, i.e. rotating, translating, reflecting the tiles *by hand* (or at the very least with the aid of dynamic geometry software, illustrating or animating the underlying transformations).

The first problem is that it is possible to draw tessellation patterns on such grids without any clear understanding of the underlying isometries by which they can be created, which in turn are conceptually important for analyzing the geometric properties embedded in the pattern, and eventually for formalizing them into proofs.

More importantly, according to Bruner this *enactive* level, where the child manipulates materials like tiles directly, is a fundamental **prerequisite** (just as in the Van

Hiele theory), for the *iconic* level, where the child now begins to deal with mental images of objects and no longer needs to manipulate them directly.

Defining and classifying

Traditionally most teachers and textbook authors have simply provided students with ready-made content (definitions, theorems, proofs, classifications, and so on) that they merely have to assimilate and regurgitate in tests and exams. Traditional geometry education of this kind can be compared to a cooking and bakery class where the teacher only shows students cakes (or even worse, only pictures of cakes) without showing them what goes into the cake and how it is made. In addition, they're not even allowed to try their own hand at baking!

Mathematicians and mathematics educators alike have often criticized the direct teaching of geometry definitions with no emphasis on the underlying process of defining. The well-known mathematician Hans Freudenthal (1973:416-418) also strongly criticized the traditional practice of the direct provision of geometry definitions as follows:

> "*... the Socratic didactician would refuse to introduce the geometrical objects by definitions, but wherever the didactic inversion prevails, deductivity starts with definitions.*
>
> *... most definitions are not preconceived but the finishing touch of the organizing activity. The child should not be deprived of this privilege ... Good geometry instruction can mean much - learning to organize a subject matter and learning what is organizing, learning to conceptualize and what is conceptualizing, learning to define and what is a definition. It means leading pupils to understand why some organization, some concept, some definition is better than another.*"

Just knowing the definition of a concept does not at all guarantee understanding of the concept. For example, although a student may have been taught, and be able to recite, the standard definition of a parallelogram as a quadrilateral with opposite sides parallel, the student may still not consider rectangles, squares and rhombi as parallelograms, since the students' concept image of a parallelogram is that not all angles or sides are allowed to be equal.

According to the Van Hiele theory understanding of formal, textbook definitions only develops at Level 3, and that the direct provision of such definitions to students at lower levels would be doomed to failure. In addition, if we take the constructivist theory of learning seriously (namely that knowledge simply cannot be transferred directly from one person to another, and that meaningful knowledge needs to be actively (re)-constructed by the learner), students ought be engaged in the activity of defining and allowed to choose their own definitions at each level. This implies allowing the following possible kinds of meaningful definitions for a rectangle at each Van Hiele level:

Van Hiele 1

Visual definitions; for example, a rectangle is a figure which looks like this (draws or identifies a quadrilateral with all angles 90° and two long and two short sides).

Van Hiele 2

Uneconomical definitions; for example, a rectangle is a quadrilateral with opposite sides parallel and equal, all angles 90°, equal diagonals, half-turn-symmetry, two axes of symmetry through opposite sides, two long and two short sides, etc.

Van Hiele 3

Correct, economical definitions; for example, a rectangle is a quadrilateral with two axes of symmetry though opposite sides.

Hierarchical versus Partition Definitions

Though children at an early age are capable of understanding class inclusions like "*cats and dogs are animals*", it appears substantially more difficult to accomplish with geometric figures. Generally, students' spontaneous definitions at Van Hiele Levels 1 and 2 as shown above would also tend to be *partitional*; in other words, they would not allow the inclusion of the squares among the rectangles (by explicitly stating two long and two short sides). In contrast, according to the Van Hiele theory, definitions at Level 3 are typically *hierarchical*, which means they allow for the inclusion of the squares among the rectangles, and would not be understood by students at lower levels.

In traditional instruction children are mostly introduced to rectangles, rhombi, parallelograms, etc. as '*static geometric objects*'. For example, a rectangle might be introduced by comparison to the shape of a door or a static picture in a book, but a door or a picture in a book cannot be transformed into a square (unless parts are cut off). So the concept rectangle is from the start introduced as a concept completely disjoint from a square. Unfortunately this partition classification schema then becomes entrenched and fossilized over time, and appears very resistant to change.

The conceptual difficulty of geometric class inclusion was already shown by Mayberry (1981) who found that only 3 out of 19 preservice mathematics teachers indicated squares also as rectangles on a sheet of some given quadrilaterals. Though valid criticism can be raised against some of questions used by Mayberry, as well as by Usiskin (1982) to evaluate hierarchical thinking, since given a set of different quadrilaterals, students might just mark the most general quadrilateral (e.g. a general parallelogram) when asked to mark it, simply *not knowing or realizing the intention* of the question was that all the special cases (e.g. rectangles, rhombi & squares) had to be marked as well.

In research conducted by De Villiers & Njisane (1987) with 4015 students from KwaZulu (South Africa) with some modified questions for evaluating hierarchical thinking (see Figure 6 for an example), some small improvement was observed. Nonetheless, it was found that very little progress occurred in their hierarchical thinking from Grade 9 to Grade 12, only ranging from 0.5% to 5.1% success with a 50% criterion on test items evaluating hierarchical thinking. This contrasts starkly with Van Hiele 3 proficiency levels in one-step and two-step deductions that respectively improved from 2.5% and 0.2% in Grade 9 to 63.3% and 42.6% in Grade 12. More recent findings by Atebe & Schäfer (2008) with a group of Nigerian and South African similarly showed that class inclusions of quadrilaterals among the investigated group from Grades 10-12 were almost completely absent.

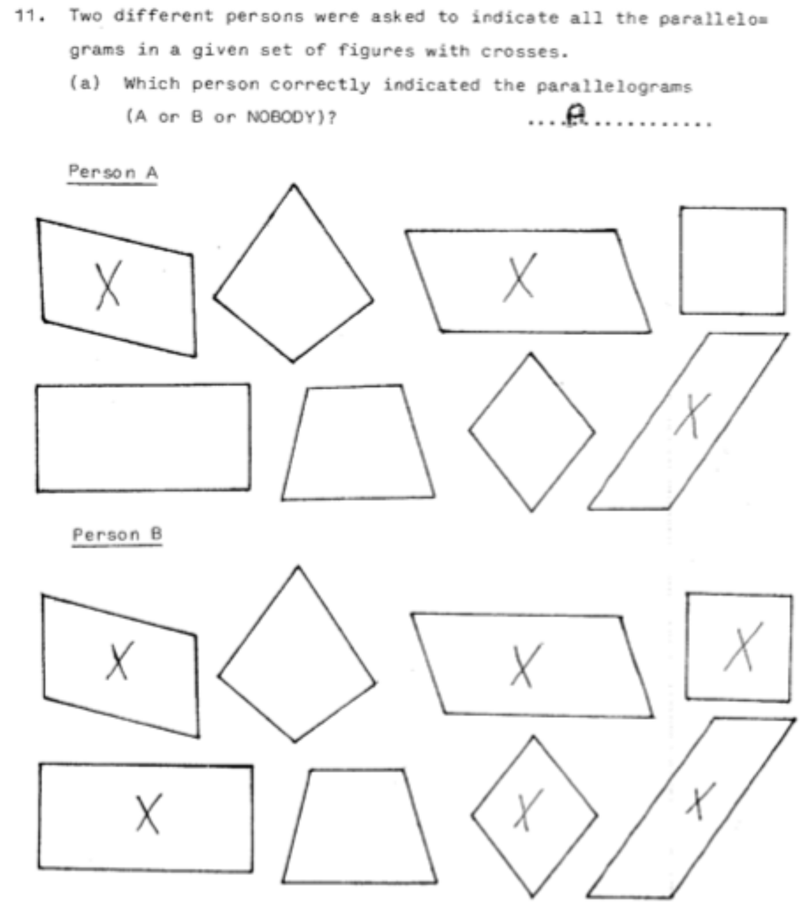


Figure 6: Testing class inclusion

Formal Van Hiele Level 3 definitions in textbooks are often preceded by an activity whereby students have to compare in tabular form various properties of the quadrilaterals, designed with the intention to assist students to see that a square, rectangle and rhombus have all the properties of a parallelogram, and that they therefore could be viewed as special cases. However, research reported in De Villiers (1994) shows that many students, even after doing tabular comparisons and other activities, if given the opportunity, still preferred to define quadrilaterals in *partitions*. (In other words, they would for example still prefer to define a parallelogram as a quadrilateral with both pairs of opposite sides parallel, but not all angles or sides equal).

For this reason, it seems that students should be allowed to formulate their own definitions irrespective of whether they are partitional or hierarchical. By then discussing and comparing in class the relative advantages and disadvantages of these two different ways of classifying and defining quadrilaterals (both of which are mathematically correct), students may be led to realize that there are certain advantages in accepting a hierarchical classification. For example, if students are asked to compare the following two definitions for the parallellograms, they might realize that the former is more **economical** than the latter:

hierarchical: A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

partitional: A parallelogram is a quadrilateral with both pairs of opposite sides parallel, but not all angles or sides equal.

Clearly, partitional definitions are longer since they have to include additional properties to ensure the exclusion of special cases. Another advantage of a hierarchical definition for a concept is that all theorems proved for that concept then automatically apply to its special cases. For example, if we prove that the diagonals of a parallelogram bisect each other, we can immediately conclude that it is also true for rectangles, rhombi and squares. If however, we classified and defined them partitionally, we would have to prove separately in each case, for parallelograms, rectangles, rhombi and squares, that their diagonals bisect each other. Clearly to reproduce all these proofs is clearly very uneconomical. It seems clear that unless the role and function of a hierarchical classification is meaningfully addressed in class, many students will have difficulty in understanding why their intuitive, partitional definitions are not used.

Engaging students in defining geometric concepts like the quadrilaterals also provide valuable opportunity for students to learn how to construct counter-examples to incomplete or wrong definitions that they may come up with. For example, to be able to show that "*a kite is a quadrilateral with perpendicular diagonals*" is an incomplete definition to finding a quadrilateral with perpendicular diagonals that is not a kite.

One common difficulty students have in producing correct counterexamples to incomplete definitions is that they often try to refute a definition with a special case. For example, for the incorrect definition "*a rectangle is any quadrilateral with congruent diagonals*," some students will provide a square as a counterexample. But obviously a square is not a valid counterexample, because a square *is* a rectangle.

Therefore, students should already have developed a sound understanding of a hierarchical (inclusive) classification of quadrilaterals before being engaged in formally defining the quadrilaterals themselves (Craine & Rubenstein, 1993; Casa & Gavin, 2009). This development can be fostered by using interactive geometry software, figures created with flexible wire, or paper-strip models of quadrilaterals. Indeed, working with a group of five Grade 6 students using flexible wire and paper strip models, Malan (1986) found that they all were eventually able to successfully make hierarchical class inclusions of the quadrilaterals. In addition, he found that the language used for describing class inclusions was an important factor (e.g. calling a square a *special* rectangle).

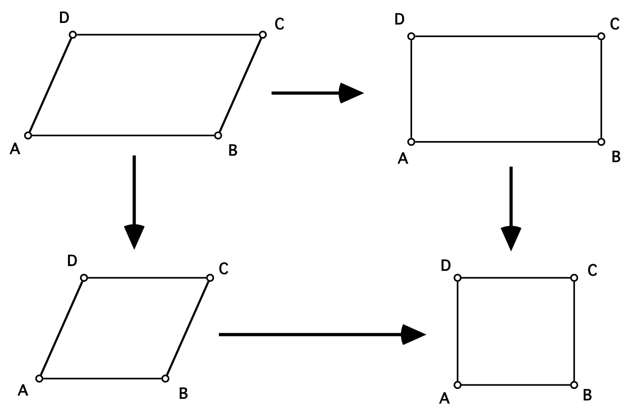


Figure 7: Dynamic transformation of parallelogram

Specifically, the dynamic nature of geometric figures constructed in dynamic software like *Sketchpad* may make the acceptance of a hierarchical classification of the quadrilaterals far easier at lower Van Hiele levels. For example, if students construct a quadrilateral with opposite sides parallel, then they will notice that they could easily drag it into the shape of a rectangle, rhombus or square as shown in Figure 7. In fact, it seems quite possible that at least some students would be able to accept and understand this even at Van Hiele Level 1 (Visualization), but further research into this particular area is needed. It is quite possible too that student difficulties with hierarchical class inclusion is largely the result of traditional instructional practices, something already observed by Mayberry (1981:8) when she wrote: "*It is conceivable that the observed levels are an artifact of the current curriculum or the instruction given to the students* …"

The author has developed an experimental Java applet at http://math.kennesaw.edu/~mdevilli/quadclassify.html where the most common quadrilaterals are not introduced via formal definition, but simply introduced visually. Through guided dragging it is envisaged that a child at Van Hiele 1 may, for example, more easily develop a *dynamic concept image* of a rectangle as one that can change into a square when all its sides become equal. Teachers and researchers are invited to try out these activities and any feedback or reports are most welcome.

Construction and Measurement

It should first be pointed out that certain kinds of construction activities (with dynamic geometry software or by pencil and paper) are inappropriate at Van Hiele Level 1. For example, someone was recently overheard at a conference commenting that she was unpleasantly dismayed at how difficult young children found the task of constructing a "*dynamic*" square with *Sketchpad*. However, if the children were still at Van Hiele Level 1, then it is not surprising at all—how can they construct a square if they do not yet know its properties (Level 2) and that some properties are sufficient and others not (that is, know the logical relationships between the properties—Level 3)?

In fact, at Van Hiele Level 1 it would appear to be far more appropriate to provide children with ready-made sketches of quadrilaterals in dynamic geometry software, which they can then easily manipulate and first investigate visually. Next, they could start using the measure features of the software to analyze the properties (and learn the appropriate terminology) to enable them to reach Level 2. Only then would it be appropriate to challenge them to construct such dynamic quadrilaterals themselves, thus assisting the transition to Level 3.

In other words, students who are predominantly at Van Hiele Level 2 cannot yet be expected to logically check their own descriptions (definitions) of quadrilaterals, but they should be allowed to do so by accurate construction and measurement. For example, students could evaluate the following attempted descriptions (definitions) for a rhombus by construction and measurement as shown in Figure 8:

(1) A rhombus is a quadrilateral with all sides equal.

(2) A rhombus is a quadrilateral with perpendicular, bisecting diagonals.

(3) A rhombus is a quadrilateral with bisecting diagonals.

(4) A rhombus is a quadrilateral with one pair of adjacent sides equal and both pairs of opposite sides parallel.

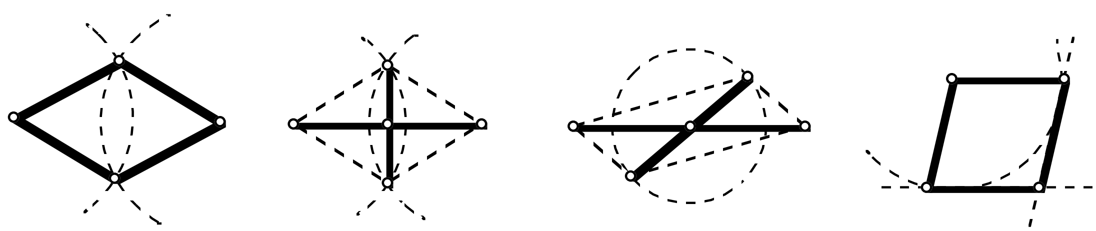


Figure 8: Construction & measurement

In the first example, students should construct a quadrilateral so that all four sides are equal, and then could notice that the diagonals always bisect each other perpendicularly, irrespective of how they drag it. This clearly shows that the property of "perpendicular bisecting diagonals" is a consequence of their constructing "all four sides equal.” On the other hand, such testing also clearly shows when a description (definition) is incomplete (contains insufficient properties), as in the third example above.

Conceptually, constructions like these are extremely important for assisting the transition from Van Hiele Level 2 to Van Hiele Level 3. It helps to develop an understanding of the difference between a *premise* and *conclusion* and their *causal* relationship; in other words, of the logical structure of an "*if-then*" statement. For example, statement 4 could be rewritten by students as: "**If** a quadrilateral has one pair of adjacent sides equal and both pairs of opposite sides parallel, **then** it is a rhombus (that is, has all sides equal, perpendicular bisecting diagonals, and so on)". Smith (1940) reported marked improvement in students' understanding of "*if-then*" statements after letting them make constructions to evaluate geometric statements as follows:

> "*Pupils saw that when they did certain things in making a figure, certain other things resulted. They learned to feel the difference in category between the relationships they* ***put*** *into a figure - the things over which they had control - and the relationships which* ***resulted*** *without any action on their part. Finally the difference in these two categories was associated with the difference between the* ***given*** *conditions and* ***conclusion****, between the if-part and the then-part of a sentence.*"

Proof Phases in Geometry Education

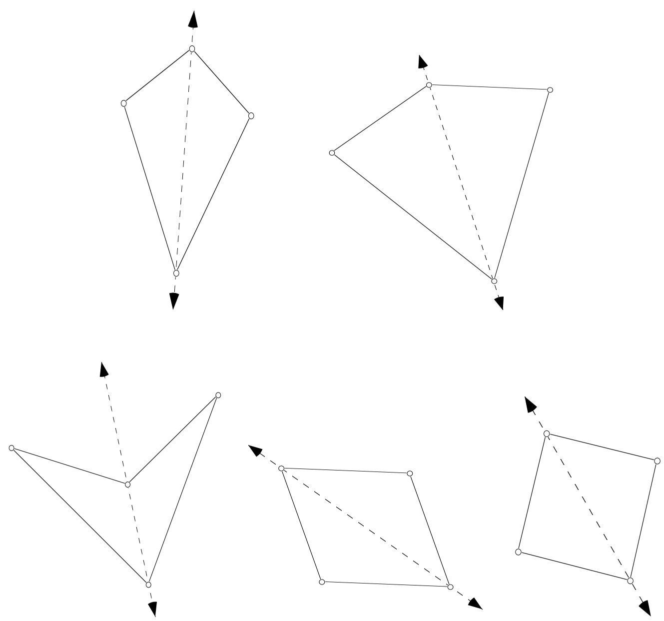
According to the Van Hiele theory, for learning to be meaningful, students should become acquainted with, and explore, geometry content in phases which correspond to the Van Hiele Levels. A serious shortcoming of the Van Hiele theory, however, is that there is no explicit distinction between different possible functions of proof. For example, the development of deductive thinking appears first within the context of *systematization* at Van Hiele Level 3 (Ordering). Empirical research by De Villiers (1991) and Mudaly & De Villiers (2000) seem to indicate, however, that the functions of proof such as *explanation*, *discovery* and *verification* can be meaningful to students outside a systematization context, in other words, at Van Hiele Levels lower than Van Hiele Level 3, provided the arguments are of an intuitive or visual nature; for example, the use of symmetry or dissection.

From experience, it also seems that a prolonged delay at Van Hiele Levels 1 and 2 before introducing proof actually makes introducing proof later as a meaningful activity even more difficult. Below are four example activities sequenced to not only correspond to the Van Hiele levels, but also to incorporate a distinction between some different functions of proof at these levels.

Activity 1: *Exploration of properties of a kite*

In this activity students use Sketchpad to first construct a kite by using reflection, and then explore its properties (for example, angles, sides, diagonals, circum circle). By dragging, students also explore special cases (rhombus, square).

- involves Van Hiele Level 1 (visualization) and Van Hiele Level 2 (analysis and formulation of properties)
- opportunity for class inclusion of rhombi and squares by dragging is provided
- properties of kite are *explained* (proved) in terms of reflective symmetry



Construct

1. Draw a line through two points, and then any point not on the line.
2. Reflect the "outside" point in the line.
3. Connect corresponding points to obtain a quadrilateral as shown above.

Investigate

1. Make conjectures regarding the following properties of the above figures:
   (a) sides
   (b) angles
   (c) diagonals

(d) inscribed or circumscribed circle

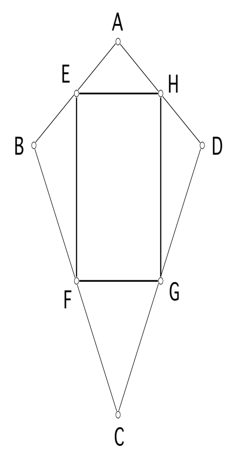
2. Can the above figure sometimes be a parallelogram, rectangle, rhombus or square?
3. Logically explain your conjectures in Question 1 in terms of symmetry.

Activity 2: *Constructing midpoints of sides of kite*

Students construct the midpoints of the sides of a dynamic kite and explore the kind of figure formed (leading to the conjecture that it is a rectangle).

- That midpoints form a rectangle is explained in terms of perpendicularity of diagonals, leading to the *discovery* that this would be true for any quad with perpendicular diagonals

Construct and connect the midpoints of the sides of a kite.



1. Investigate the type of quadrilateral formed by the midpoints of its sides.
2. Logically explain your conjecture.
3. From Question 2, can you find/construct another more general type of quadrilateral that will have the same midpoint property?

   (The result generalizes to any quadrilateral with perpendicular diagonals).

Activity 3: *Describing (defining) a kite*

Students select different subsets of the properties of a kite as possible descriptions (definitions), and first check whether they are necessary and sufficient by using them in a Sketchpad construction, and then by logical reasoning (proof).

- involves Van Hiele Level 3 (local ordering)
- the explicit function of proof here is *systematization* (that is, the deductive organization of the properties of a kite).
- involves the mathematical process of *descriptive* defining

The *kite* has the following properties:

a. (At least) one line of symmetry through a pair of opposite angles
b. Perpendicular diagonals (with at least one bisecting the other)
c. (At least) one pair of opposite angles equal
d. Two (distinct) pairs of adjacent sides equal
e. (At least) one diagonal bisecting a pair of opposite angles
f. Incircle

1. How would you over the phone explain what "kites" are to someone not yet acquainted with them? (Try and keep your description as short as possible, but ensure that the person has enough information to make a correct drawing of the quadrilateral).
2. Try formulating two alternative descriptions. Which of the three do you like best? Why?

Results from Govender & De Villiers (2003), De Villiers (1998, 2004) and Sáenz-Ludlow & Athanasopoulou (2007) do indicate some improvement and positive gains in student understanding of the nature of definitions, as well as in their ability to define geometric concepts such as quadrilaterals themselves.

Activity 4: *Generalizing or specializing a kite*

Students generalize by leaving out some of the properties and specialize by adding more properties. The properties of the newly defined objects are then explored by construction on Sketchpad and/or by deductive reasoning.

- involves Van Hiele Level 4 (global ordering)
- involves the mathematical process of *constructive* defining

1. Generalize the concept "*kite*" in different ways by leaving out, altering or generalizing some of its properties.

   (One possibility is to generalize to a 2n-gon; ie. a polygon with at least one axis of symmetry through a pair of opposite angles. Other possibilities are to generalize to a quadrilateral with at least one pair of adjacent sides equal, to one with one diagonal bisected by the other, or to one circumscribed around a circle - a circum quad).

2. Specialize the concept "*kite*" in different ways by the addition of more properties.

   (Possibilities to consider are a kite inscribed in a circle, a kite with at least three equal angles, or a kite with another axis of symmetry through a pair of opposite angles - a rhombus).

The briefly-described activities above are intended as examples of how students can be engaged in proof at levels lower than Van Hiele Level 3. Examples of more fully developed proof and defining activities are available in De Villiers (2003 & 2009). Defining quadrilaterals also provide an excellent context for introducing and developing students' understanding of necessary and sufficient conditions as discussed and illustrated in De Villiers et al (2009). The study by Atebe & Schäfer (2008), for example, showed how students frequently mistook necessary for sufficient conditions in the context of quadrilaterals.

Van Hiele levels in other areas

The purpose of this section is to give a few examples of the application of the general Van Hiele levels of thinking to other areas in mathematics as suggested by Hoffer (1981).

Analogous to the Van Hiele model for geometry (Van Hiele, 1973), the following conjectured thought levels for Boolean Algebra evolved from an experimental course for Boolean Algebra that was developed in the period 1978 -1985 (De Villiers, 1986).

Level 1 (Interpretation & representation of switching circuits)

Children are able to connect switches in series and/or parallel circuits from given diagrammatic representations, and vice versa. They are, however, not yet able to discern nor symbolize any of the properties of switching circuits like the distributive property, De Morgan's laws, absorption laws, etc.

Level 2 (Analysis of switching properties)

Children now begin to analyze and become conscious of the various structural properties of switching circuits, and are able to verbalize or symbolize them. They are able to apply these properties in solving practical properties such as simplifying switching circuits or designing new ones. However, at this level they are still unable to see the logical relationships (implications) between the various properties. The meaning of proof is one of *justification*, e.g. checking the validity of 'non-obvious' properties like $f + \bar{f} \cdot g = f + g$ by experimental verification or the consideration of all possibilities through the completion of truth tables.

Level 3 (Logical Implication: deduction)

Now children become aware of the logical relationships between the various switching properties; in other words, that certain properties may be derived from others. The ability to do deductive proofs, to axiomatize and systematize and the development of an understanding of the significance of axioms, characterizes this level. Proof now assumes new meaning, namely that of *systematization* of mathematical statements into a logical system.

Through investigations of the development of students' language for describing functions and the history of the description of motion and the development of calculus, Isoda (1996) has proposed the following thought levels of 'language about functions'.

Level 1 (Level of Everyday Language)

Students describe relations in phenomena using everyday language obscurely. They can discuss changes in numbers using calculations, but usually their descriptions are done with or focused on one physically evident variable, the *dependent variable*. Even if they are aware of co-variation, it is difficult for them to explain it appropriately using two variables because their descriptions of relations are done obscurely, using everyday language. So it is difficult for them to compare different phenomena at once, appropriately.

Level 2 (Level of Arithmetic)

Students describe the rules of relations using tables. They make and explore tables with arithmetic. Their descriptions of relations in phenomena are more precise with tables than with the only everyday language of Level 1. Students have general concepts about some rules of relations, for instance, proportion. Students can compare different phenomena using such rules. They describe rules of relations as co-variation and when reading tables, their interpretation of the co-variation of variables is at least as strong as their interpretation of correspondence. Students can use formulas and graphs to represent rules and relations too, but it is not easy for them to translate between notations.

Level 3 (Level of Algebra and Geometry)

Students describe functions using equations and graphs. To explore function, they translate among the notations of tables, equations and graphs and use algebra and geometry. At this level, their notion of function, which they already understand well, involves the representation of different notations already integrated as the mental image. For example, they can easily find the equation emerging from the graph, and the graph from the equation.

Level 4 (Level of Calculus)

Students describe and investigate the behaviour of functions using calculus. In calculus, functions are described in terms of *derived* or *primitive* functions. For example, to describe the features of a function we use its derived function (derivative), which is already learned. The theory of calculus is a generalized theory of this type of description.

Level 5 (Level of Analysis)

An example of language for description is functional analysis, which is a meta-theory of calculus. This level's justification is based on historical development and not yet investigated.

The following Van Hiele levels of understanding has been conjectured for trigonometry by myself and a master's student, Janeeshla Jugmohan, by way of analogy from the levels for geometry to serve as a possible learning trajectory.

Level 1 (Visualisation)

The learner is able to recognize a right triangle in different orientations and distinguish between scalene and isosceles right triangles, and identify a right triangle in the unit circle. Ability is shown to correctly recognize the 'opposite', 'adjacent' and 'rectangular' sides in various orientations, as well as the hypotenuse of a right triangle.

Level 2 (Analysis)

The learner now realizes that in a right triangle, for a fixed angle, the ratios between any two sides remain the same, irrespective of the size of the triangle (which is the concept of *similarity*). Conversely, understanding of the inverse function develops, for example $\sin x = \frac{1}{2} \;\therefore x = ?$ Students are now able to begin solving some practical and theoretical problems related to right triangles using trigonometric ratios.

Level 3 (Primitive Definition)

The aforementioned discoveries now become formalised as definitions in terms of ratios of sides of right triangles. Learners develop understanding of the increasing or decreasing nature of the trigonometric functions for angles up to 180 degrees, as well as of the associated inverse functions.

Level 4 (Circle Definition)

Understanding of the abstract definition of trigonometry as a function in the domain of the real numbers develops in terms of the unit circle. The unit circle definition defines trigonometric functions in terms of coordinates x and y, and essentially become independent of the right triangle. Learners develop understanding of the *periodicity* and the *graphical* representation of trigonometric functions, as well as of trigonometric *identities* and the ability to prove them.

Level 5 (Spherical Trigonometry, etc.)

The progression can be extended further to spherical trigonometry, which is the trigonometry on the sphere, and could be extended to other surfaces, hyperbolic functions, for example, *sinh*, *cosh*, as well as the analytic treatment of the trigonometric functions such as transcendental functions, etc. This may include the definitions of trigonometric functions as different infinite series, and to the application of Fourier and other series to different parts of mathematics and physics.

Land (1990) considered the application of the Van Hiele theory to the teaching of exponential and logarithmic functions, and Nixon (2002) analogously conjectured levels of learning and understanding sequences and series. From an analysis of the historical development of abstract algebra, Nixon (2005) conjectured the following possible spiral learning trajectory for abstract algebra:

Level 1 (Perceptual)

The focus is on learning different methods for solving certain types of equations, e.g. linear, quadratic & cubic polynomials by the balance algorithm, factorization, completion of the square, etc.

Level 2 (Conceptual)

Here there is a shift away from analyzing individual solution methods towards considering the relations or transformations between them. For example, realizing that all solution methods for polynomials involve algebraically reducing the original equation and considering the various ways it can be permuted. Complex algebra, the fundamental theorem of algebra and the work of Gauss on the composition of forms are accessible to students at this level.

Level 3 (Abstract)

At this level, the group concept can be introduced as a way of understanding and organizing the symmetries of a polynomial, and of distinguishing which polynomial equations are solvable algebraically. Rings and fields can now be introduced as further extensions on ways of investigating and comparing different algebraic structures.

Gutierrez, Pegg & Lawrie (2004) also developed characterizations of students' Van Hiele thinking levels in three-dimensional geometry to analyze the answers of 299 students in Grades 7 to 11 (aged 12 to 17 years) in New South Wales (Australia). Interestingly, some students had difficulty seeing a cube as a rectangular prism in much the same way they have difficulty perceiving a square as a rectangle. More-over, several confused a statement with its converse, and very few were able to produce formal proofs.

Concluding comments

The hierarchical, fixed order of progression through the Van Hiele levels (i.e. a pupil cannot be at level *n* without having passed through level *n-1*) have been statistically confirmed using Guttman analysis by several studies, for example, Mayberry (1981), Usiskin (1982) and De Villiers (1987). A comparative study by Smith & De Villiers (1989) of the Usiskin (1982) test and the University of Stellenbosch (1984) test further confirmed not only the hierarchical nature of the first three levels, but indicated that better classifications of students' thinking levels were obtained when more varied questions and more 'open-ended' items are used.

Pegg & Davey (1989) did a comparative study of the Van Hiele theory and the SOLO Taxonomy and found the descriptors of the latter more accurately described the geometric thinking levels of students. It is, however, still an open moot point whether such an achieved gain in 'accuracy' is worthwhile with the increased complexity of the Solo Taxonomy.

More research needs to be done on how using dynamic geometry software can enhance, or perhaps even impair, the development of geometric thinking. The use of dynamic geometry software with an experimental group of Malaysian students is reported in Idris (2009) to have contributed to them achieving higher Van Hiele levels than a control group who were taught traditionally without access to dynamic geometry.

A main concern of geometry education around the world is the continued poor level of geometric thinking among teachers themselves, and until this problem is adequately addressed, very little progress in the quality of geometry instruction is likely to be achieved. For example, Van Putten (2008) found in a post-test, that only 45% of pre-service FET (grades 10-12) teachers had reached Van Hiele Level 3 (though there was significant improvement from the pre-test).

Traditionally, the development of 'proof ability' is seen to occur from Van Hiele 3 Level onwards. Moreover, the Van Hiele model sees proof mainly as a means of 'verification', and it remains an open research question whether or not other functions of proof such as 'explanation' can be utilized and developed earlier at the visual and analytic Levels 1 and 2 respectively (see for example Mudaly & De Villiers, 2000; De Villiers, 2004). Can more explanatory visual-dissection proofs and arguments by symmetry (line, rotational, point) be developed and understood earlier by children?

Lastly, it seems one of the major outstanding research problems on the Van Hiele theory is the issue of hierarchical thinking (class inclusions). Is partition thinking the consequence of traditional geometry teaching strategies, and could hierarchical thinking be developed earlier at Van Hiele levels 1 and 2 through various strategies and using tools such as dynamic geometry software?

References

Atebe, H.U. & Schäfer, M. (2008). "As soon as the four sides are all equal, then the angles must be 90°". Children's misconceptions in geometry. *African Journal of Research in Science, Mathematics & Technology Education*, 12(2), 47-66.

Bruner, J.S. (1966). *Towards a Theory of Instruction*. New York: Norton.

Burger, W.F. & Shaughnessy, J.M. (1986). Characterizing the Van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, 17(1), 31-48.

Casa, T.M. & Gavin, M.K. (2009). Advancing Elementary School Students' understanding of quadrilaterals. In Craine, T. & Rubenstein, R. (2009). *Understanding Geometry for a Changing World*. 71st Yearbook, Reston, VA: NCTM, pp. 205-219.

Craine, T.V. & Rubenstein, R.N. (1993). A Quadrilateral Hierarchy to Facilitate Learning in Geometry. *Mathematics Teacher* 86 (January), 30–36.

De Villiers, M.D. (1986). *Boolean Algebra at school*. University of Stellenbosch: RUMEUS. (Republished & now available for purchase as PDF download or in printed book form at http://www.lulu.com/product/paperback/boolean-algebra-at-school-vol-1/14244188).

De Villiers, M.D. & Njisane (1987). The Development of Geometric Thinking among Black High School Pupils in KwaZulu (R.S.A.), *Proceedings of the Eleventh PME-conference*, Montreal: Vol.3, pp.117-123, July 1987.

De Villiers, M.D. (1987). *Research evidence on hierarchical thinking, teaching strategies and the Van Hiele Theory: Some critical comments*. University of Stellenbosch: RUMEUS. (Available from: http://math.kennesaw.edu/~mdevilli/VanHieleCritique-87.pdf)

De Villiers, M.D. (1991). Pupils' needs for conviction and explanation within the context of geometry. *Pythagoras*, 26, 18-27. (Available from: http://mzone.mweb.co.za/residents/profmd/needs.pdf)

De Villiers, M.D. (1994). The role and function of a hierarchical classification of the quadrilaterals. *For the Learning of Mathematics*, 14(1), 11-18. (Available from http://mzone.mweb.co.za/residents/profmd/classify.pdf)

De Villiers, M.D. (1998). To teach definitions or teach to define? In Olivier, A. & Newstead, K. (Eds). *Proceedings of PME 22*, Vol 2, 248-255. (Available from http://mzone.mweb.co.za/residents/profmd/define.htm)

De Villiers, M.D. (2003). *Rethinking Proof with Sketchpad 4*. Key Curriculum Press, USA.

De Villiers, M. (2004). Using dynamic geometry to expand mathematics teachers' understanding of proof. *The International Journal of Mathematical Education in Science & Technology,* 35(5), 703-724. (Available from http://mysite.mweb.co.za/residents/profmd/vanhiele.pdf)

De Villiers, M. (2009). *Some Adventures in Euclidean Geometry*. Lulu Publishers. (Available for purchase as PDF from: http://www.lulu.com/content/7622884

De Villiers, M.; Govender, R. & Patterson, N. (2009). Defining in Geometry. In Craine, T. & Rubenstein, R. (2009). *Understanding Geometry for a Changing World*. 71st Yearbook, Reston, VA: NCTM, pp. 189-203.

Feza, N. & Webb. P. (2005). Assessment standards, Van Hiele levels, and grade seven learners' understandings of geometry. *Pythagoras*, 62 (December), 36-47.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.

Fuys, D., Geddes, D. & Tischler, R. (1988). The Van Hiele Model of Thinking in Geometry among Adolescents. *JRME Monograph No. 3*, NCTM.

Govender, M. (1995). Pupils' proof-writing achievement in circle geometry. Unpublished B.Ed. dissertation, University of Durban-Westville.

Govender, R. & De Villiers, M. (2003). Constructive Evaluation of Definitions in a Dynamic Geometry Context. (2003). *Journal of the Korea Society of Mathematical Education*, 7(1), March, 41-58. (Available online at http://mathnet.kaist.ac.kr/mathnet/kms_tex/980829.pdf).

Gutierrez, A., Pegg, J. & Lawrie, C. (2004). Characterization of students' reasoning and proving abilities in 3-dimensional geometry. *Proceedings of the 28th PME Conference*, Vol. 2, pp. 511-518.

Hoffer, Alan. (1981). Geometry is More Than Proof. *Mathematics Teacher*, 74 (January): 11- 18.

Idris, N. (2009). The impact of using Geometer's Sketchpad on Malaysian students' achievement and Van Hiele geometric thinking. *Journal of Mathematics Education*, Dec, Vol. 2, No. 2, pp. 94-107.

Isoda, M. (1996). The development of the language of function: An application of Van Hiele's levels. In Puig, L. & Gutierrez, A. *Proceedings of PME 20*, Vol. 3, 105-112.

Land, J.E. (1990). Appropriateness of the Van Hiele model for describing students' cognitive processes on algebra tasks as typified by college students' learning. Ann Arbor, Mi. UMI Dissertation Services.

Mayberry, J.W. (1981). An Investigation of the Van Hiele levels of Geometric Thought in Undergraduate Preservice Teachers. Unpublished doctoral dissertation, Univ. of Georgia, Athens.

Malan, F.R.P. (1986). Onderrigstrategieë vir die oorgang van partisie denke na hierargiese denke in die klassifikasie van vierhoeke: enkele gevallestudies. [Teaching strategies for the transition of partition thinking to hierarchical thinking in the classification of quadrilaterals.] (Internal report no. 3). Stellenbosch: University of Stellenbosch, Research Unit of Mathematics Education (RUMEUS).

Mudaly, V. & De Villiers, M. (2000). Learners' needs for conviction and explanation within the context of dynamic geometry. Pythagoras, 52 (August). 20-23. (Available from http://mzone.mweb.co.za/residents/profmd/vim.pdf)

Nixon, E.G. (2002). An investigation of the influence of visualization, exploring patterns and generalization on thinking levels in the formation of concepts of sequences and series. Unpublished Masters thesis. Pretoria: UNISA.

Nixon, E.G. (2005). Creating and learning abstract algebra: Historical phases and conceptual levels. Unpublished DPhil Dissertation. Pretoria: UNISA.

Nohda, N. (1992). Geometry teaching in Japanese school mathematics. *Pythagoras*, 28, April, 18-25.

Pegg, J., & Davey, G. (1989). Clarifying level descriptors for children's understanding of some basic 2-D geometric shapes. *Mathematics Education Research Journal* 1 (1): 16- 27.

Sáenz-Ludlow, A. & Athanasopoulou, A. 2007. Investigating properties of isosceles trapezoids with the GSP: The case of a pre-service teacher. In Pugalee, D; Rogerson, A & Schinck, A, (Editors). *Proceedings of the 9th International Conference: Mathematics Education in a Global Community*, University of North-Carolina, September 7-12, 2007, pp. 577-582. (Available from: http://math.unipa.it/~grim/21_project/21_charlotte_SaenzLudlow-AthanasopoulouPaperEdit.pdf)

Smith, E. & De Villiers, M. (1989). A comparative study of two Van Hiele testing instruments. Poster presented at the 13th International Conference of PME, Paris. (Available from: http://math.kennesaw.edu/~mdevilli/VanHiele-test-comparison.pdf)

Smith, R. R. (1940). Three major difficulties in the learning of demonstrative geometry. *The Mathematics Teacher*, 33, 99-134, 150-178.

The Mathematical Association of South Africa. (1978). *South African Mathematics Project: Syllabus Proposals*. Pretoria: MASA. (Now Wits: AMESA).

University of Stellenbosch. (1984). *Levels of thinking in geometry: Tests 1 & 2*. Unit for Research in mathematics Education, Researchers: P.G. Human & M.D. de Villiers. (Respectively available at:
http://www.scribd.com/Geometry-Levels-of-Thinking-Test-1-1984/d/30882207
http://www.scribd.com/doc/30882101/Geometry-Levels-of-Thinking-Test-2-1984

Usiskin, Z. (1982). *Van Hiele levels and Achievement in Secondary School Geometry*. Final report of the CDASSG Project. Chicago: Univ. of Chicago.

Van Hiele, P.M. (1973). *Begrip & Inzicht*. Muusses: Purmerend.

Van Putten, S. (2008). Levels of Thought in Geometry of Pre-service Mathematics Educators according to the van Hiele Model. Unpublished Master's thesis, University of Pretoria. (Retrieved 26 May 2010 from: http://upetd.up.ac.za/thesis/available/etd-05202008-130804/unrestricted/dissertation.pdf)

Wood, D., Bruner, J.S., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Psychology and Psychiatry*. 17.

Wirszup, I. (1976). Breakthroughs in the Psychology of Learning and Teaching Geometry. In J.L. Martin (Ed.). *Space and Geometry*. Columbus, Ohio: Eric Centre.

Wu, D. & Ma, H. (2006). The distribution of Van Hiele levels of geometric thinking among 1st through 6th graders. In Novotna, J. et al, *Proceedings of PME 30*, Vol 5, pp. 409-416. Prague: PME.