Published in *Euclides*, December 2012, no. 3, pp. 133. Journal of the Nederlandse Vereniging van Wiskundeleraars, http://www.nvvw.nl

Another proof of De Opgave 2011:
An application of a Van Aubel generalization

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I found the problem circulated by Dick Klingens at the NVvW annual meeting in November 2011 very interesting, as well as the wide variety of solutions that were produced for this problem as published in the March 2012 issue of *Euclides* (Lecluse, 2012).

Here is another relatively simple proof to the result that uses the nine-point circle, and then also shows it as a special case of a generalization of Van Aubel's theorem.

Let's first start by recapping two equivalent formulations of the result:

1) Two circles K_1 and K_2 intersect each other in B and Q. A line through Q cuts K_1 in point A and K_2 in point C. The points E and F are the respective midpoints of the arcs BA and BC that do not contain Q. If M is the midpoint of AC, then angle $EMF = 90^\circ$.

2) Given two isosceles triangles ABC and BDE with respective bases AB and BD lying in a straight line (and on the same side of the line), and BC is perpendicular to BE. If M is the midpoint of AD, then angle $CME = 90^\circ$.

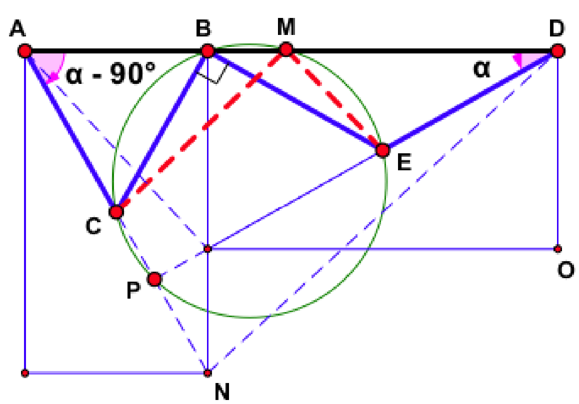

Figure 1

Let us first focus on proving the second version as shown in Figure 1. Construct similar rectangles ABN and ODB as shown from the given isosceles triangles ABC and BDE.

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Then construct the circumcircle of *BCE* to intersect *AD* in *M* (which we will prove further down, is the midpoint of *AD*). Since angle *CBE* = 90º, it follows that *CME* = 90º (angles on same diameter *CE*). Angle *APD* = 90º from the angle sum in triangle *APD*. Hence, it follows that *P* also lies on the circle *BCE* as the angle *APD* is subtended by the diameter *CE*. Therefore in triangle *AND*, point *P* is the foot of the altitude from *D* to *AN*. Since *NB* is the altitude from *N* to *AD* in the same triangle, and *C* is the midpoint of *AN*, it follows that circle *BCE* is the nine-point circle of triangle *AND*. Thus, the other intersection point *M* of the nine-point circle with side *AD* is the midpoint of *AD*, and completes the proof.

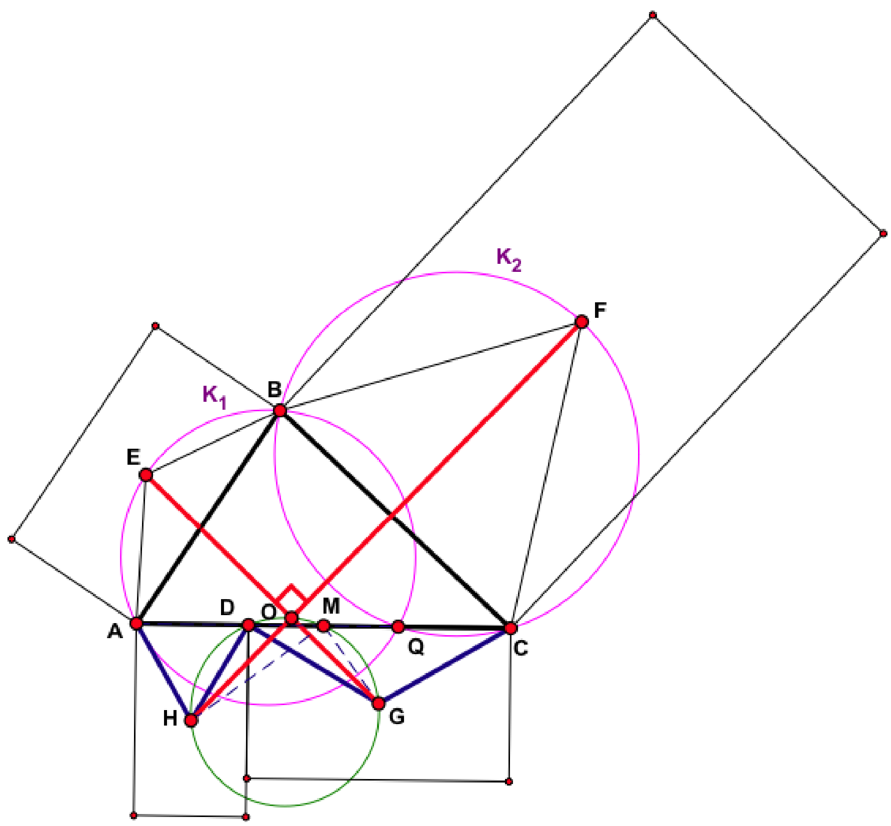

Figure 2

Perhaps more interestingly, is that the result is a special case of the following generalization of Van Aubel's theorem proved in De Villiers (1998): if similar rectangles *E*, *F*, *G* and *H* are constructed in alternating order (orientation) on the sides of any quadrilateral *ABCD*, then the lines connecting the centres of the rectangles on the opposite sides of *ABCD* are perpendicular to each other. For example, consider Figure 2, which shows sides *CD* and *DA* of the quadrilateral *ABCD* in a straight line. From this Van Aubel generalization, it then follows that *EG* and *FH* are perpendicular in the point *O*, which therefore also lies on the circle *HDMG* (from the previous result).

Also note that formulation 1 of the result is now quite nicely illustrated in the top part of Figure 2 by the circles K_1 and K_2 intersecting in *B* and *Q*, the straight line *AQC*, where in this case we have *EOMQF* concyclic.

The reader is now lastly invited to explore Figure 2 interactively at: http://dynamicmathematicslearning.com/vanaubel-application1.html

References

De Villiers, M. (1998). Dual generalizations of Van Aubel's theorem. *The Mathematical Gazette*, Nov, 405-412. (Available to download from: http://mzone.mweb.co.za/residents/profmd/aubel2.pdf)

Lecluse, T. (2012). Vanuit de oude doos: De Opgave 2011, uitgedeeld op de Jaarvergadering. *Euclides*, *87(5)*, Maart, 215-217. (This paper with 5 different proofs can be downloaded from: http://dynamicmathematicslearning.com/lecluse-opgave2011.pdf Nine other proofs can also be downloaded directly from the NVvW website in ZIP-format (2.8 Mb) at: http://www.nvvw.nl/media/downloads/najaar2011.zip)