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A unifying generalization of Turnbull's theorem

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This article presents a generalization of a familiar result from elementary geometry for cyclic quadrilaterals, as well as a generalized dual for circumscribed quadrilaterals. It can be utilized in the classroom to illustrate how new mathematics is sometimes discovered or created.

A mathematical theory, like any other scientific theory, is a social product. It is created and developed by the dialectical interplay of many minds, not just one mind. When we study the history of mathematics, we do not find a mere accumulation of new definitions, new techniques, and new theorems. Instead, we find a repeated refinement and sharpening of old concepts and old formulations, a gradually rising standard of rigour and an impressive secular increase in generality and depth. Each generation of mathematicians rethinks the mathematics of the previous generation, discarding what was faddish or superficial or false and recasting what is still fertile into new and sharper forms. Nicolas Goodman [1].

1. Introduction

In a problem section in [2] readers were informed about the classroom discovery by a student named Turnbull [3], namely that the sum of the alternate angles of a convex cyclic hexagon are equal to 360°, and asked to prove it.

This led to a subsequent response by De Villiers [4] in which the following generalization of this result, and of the related result for cyclic quadrilaterals, was given:

(1) If $A_1A_2...A_{2n}$ $(n>1)$ is any convex cyclic polygon P then the two sums of alternate interior angles of P are each equal to $(n-1)\pi$.

In the same article it was also pointed out that this result has the following interesting dual:

(2) If $A_1A_2...A_{2n}$ $(n>1)$ is any convex circumscribed polygon P then the two sums of alternate sides of P are equal.

This was followed up by Klamkin [5] in an article in which he pointed out that the following analogous result exists for cyclic star $4n$-gons:

(3) If $A_1A_2...A_{4n}$ $(n>1)$ is any cyclic star polygon P in which each vertex A_i is joined to vertex A_{i+2n-1}, then the two sums of alternate interior angles of P are each equal to π.

De Villiers [6] then consequently pointed out that a dual also exists for the latter case, for example:

(4) If $A_1A_2 \ldots A_{4n}$ $(n>1)$ is any circumscribed star polygon P in which each vertex A_i is joined to vertex A_{i+2n-1}, then the two sums of alternate sides of P are equal.

What follows, is a brief description of a more comprehensive investigation of Turnbull's theorem which led to the discovery of a further, unifying generalization.

2. A unifying generalization

Having previously in [7] explored a classification of what the author called 'generalised poly-figures', and in which the cyclic star polygons referred to in version 3 are only some of the possible cases that can be generalized from convex cyclic polygons (or more generally from simple closed polygons) in terms of their interior angle sum, he intuitively sensed that other generalizations (and in fact a unifying generalization) of Turnbull's theorem would exist. After some thought he soon constructed a class of cyclic $2n$-gons $(n>1)$ with the connecting rule $A_i \rightarrow A_{i+n-1}$ as shown in Figure 1, for which the total interior angle sum is 2π and the two sums of alternate interior angles are therefore equal to π. (Note that the cases $n=4, 6, 8, 10, \ldots$ are the same polygons referred to in version 3 above.) The cases for $n=3, 5, 7\ldots$ can also be seen to consist of a set of two figures, i.e. triangles (a generalized star of David), star pentagons, star septagons, etc., respectively overlapping in such a manner that A_1; A_3; A_5; $\ldots$ A_{2n-1} belong to the one figure and A_2; A_4; A_6; $\ldots$ A_{2n} belong to the other figure. Note therefore that in these cases the polygons need not be cyclic for the result to be true.

Next he constructed a class of cyclic $2n$-gons $(n>2)$ with the connecting rule $A_i \rightarrow A_{i+n-2}$ as shown in Figure 2, for which the total interior angle sum is 4π and the two sums of alternate interior angles are therefore equal to 2π. Here the cases for $n=4$ and $n=6$ can also be seen to consist respectively of two overlapping quadrilaterals and two overlapping generalized stars of David, and therefore need not be cyclic.

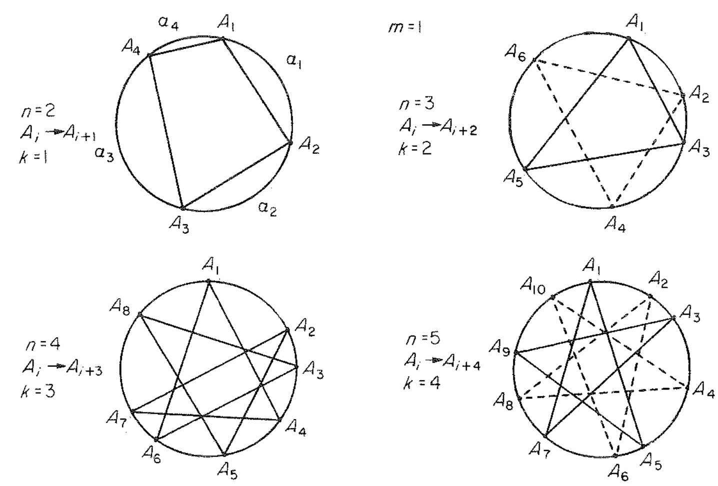

Figure 1.

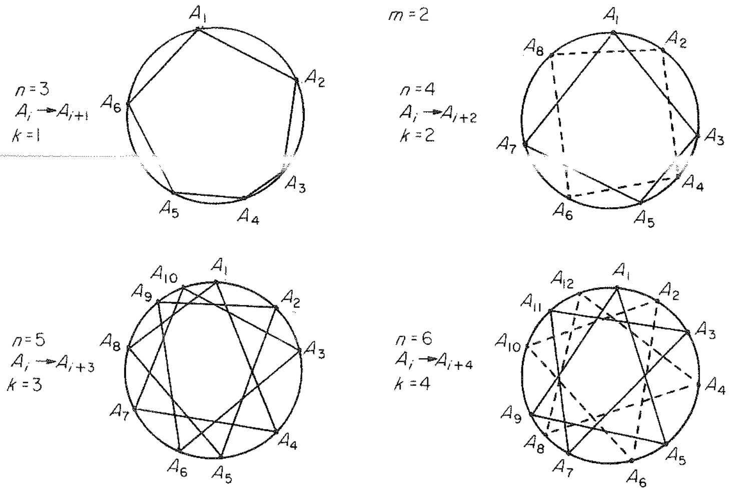

Figure 2.

In a similar fashion other general classes can also be formed from convex cyclic octagons, decagons, etc. The connecting rule in general would then simply be $A_i \rightarrow A_{i+n-m}$ with $m=1, 2, 3, \ldots$ and $n-m \geqslant 1$. Or alternatively and more simply, the general connecting rule can be formulated as $A_i \rightarrow A_{i+k}$ where $k=1, 2, 3, \ldots$ is the *total turning* (the number of complete rotations of 2π) one would undergo walking completely around the perimeter of each figure. We can therefore now formulate the following beautiful generalization of Turnbull's theorem which includes all the previous cases[1]:

Theorem 1. If $A_1 A_2 \ldots A_{2n}$ $(n>1)$ is any cyclic $2n$-gon in which vertex $A_i \rightarrow A_{i+k}$ (vertex A_i is joined to A_{i+k}), then the two sums of alternate interior angles are each equal to $m\pi$ (where $m=n-k$).

Although generalization (1) can easily be proved by mathematical induction from the special case for cyclic quadrilaterals and considering the addition of two vertices at a time, the following proof, based on the notation and approach in [5], proved more convenient for the unifying generalization.

Proof. Let the measure of the minor arc $A_i A_{i+1} = \alpha_i$; $i=1, 2, \ldots, 2n$ with $\alpha_{2n+1} = \alpha_1$. (See first figure in Figure 1.)

First consider the case for $m=1$, then

$$2\hat{A}_1 = \alpha_n + \alpha_{n+1}$$

$$2\hat{A}_3 = \alpha_{n+2} + \alpha_{n+3}$$

$$\vdots$$

$$2\hat{A}_{2n-1} = \alpha_{n-2} + \alpha_{n-1}$$

Then by addition we have

$$2\sum_{i=1}^{n} \hat{A}_{2i-1} = 2\pi \Leftrightarrow \sum_{i=1}^{n} \hat{A}_{2i-1} = \pi$$

In the same manner we can prove that the sum of the even alternate interior angles is also π. (Or alternatively we can in general find the sum of the even alternate interior angles simply by subtracting πm from the total interior angle sum of these $2n$-gons, namely $S=2\pi m$. This follows directly as a special case from a proof for n-gons in general given in [7]).

After also considering the case for $m=2$, the general argument was formulated as follows:

$$\begin{aligned}
2\hat{A}_1 &= \alpha_{n-m+1}+\alpha_{n-m+2}+\ldots+\alpha_{n+m-1}+\alpha_{n+m}\\
2\hat{A}_3 &= \alpha_{n-m+3}+\alpha_{n-m+4}+\ldots+\alpha_{n+m+1}+\alpha_{n+m+2}\\
&\vdots\\
2\hat{A}_{2n-1} &= \alpha_{n-m-1}+\alpha_{n-m}\quad+\ldots+\alpha_{n+m-3}+\alpha_{n+m-2}
\end{aligned}$$

Then by addition and rearrangement we have:

$$\begin{aligned}
2\sum_{i=1}^{n}\hat{A}_{2i-1} &= m(\alpha_{n+m}+\alpha_{n+m+1}+\ldots+\alpha_{n+m-2}+\alpha_{n+m-1})\\
&= m(2\pi)\\
\Leftrightarrow \sum_{i=1}^{n}\hat{A}_{2i-1} &= m\pi
\end{aligned}$$

Or alternatively, mathematical induction can be used in an informative manner as follows. As it is true for $m=1$ as already shown, let's assume that it is true for $m=p$, and therefore that the following is true:

$$\begin{aligned}
2\hat{A}_1 &= \alpha_{n-p+1}+\alpha_{n-p+2}+\ldots+\alpha_{n+p-1}+\alpha_{n+p}\\
2\hat{A}_3 &= \alpha_{n-p+3}+\alpha_{n-p+4}+\ldots+\alpha_{n+p+1}+\alpha_{n+p+2}\\
&\vdots\\
2\hat{A}_{2n-1} &= \alpha_{n-p-1}+\alpha_{n-p}\quad+\ldots+\alpha_{n+p-3}+\alpha_{n+p-2}
\end{aligned}$$

$$\Rightarrow 2\sum_{i=1}^{n}\hat{A}_{2i-1}=p(2\pi)\Leftrightarrow\sum_{i=1}^{n}\hat{A}_{2i-1}=p\pi$$

Now consider $m=p+1$, then:

$$\begin{array}{rl|l|l}
2\hat{A}_1 = & \alpha_{n-p}+ & \alpha_{n-p+1}+\ldots+\alpha_{n+p} & +\alpha_{n+p+1}\\
2\hat{A}_3 = & \alpha_{n-p+2}+ & \alpha_{n-p+3}+\ldots+\alpha_{n+p+2} & +\alpha_{n+p+3}\\
\vdots & & & \\
2\hat{A}_{2n-1} = & \alpha_{n-p-2}+ & \alpha_{n-p-1}+\ldots+\alpha_{n+p-2} & +\alpha_{n+p-1}
\end{array}$$

By addition, rearrangement and utilization of the assumed truth for $m=p$ we then obtain

$$2\sum_{i=1}^{n}\hat{A}_{2i-1}=p(2\pi)+2\pi\Leftrightarrow\sum_{i=1}^{n}\hat{A}_{2i-1}=(p+1)\pi$$

showing that it is true for $m=p+1$. But the argument is true for $m=1$, and therefore according to the principle of mathematical induction, it would be true for all $m=1,2,3\ldots$.

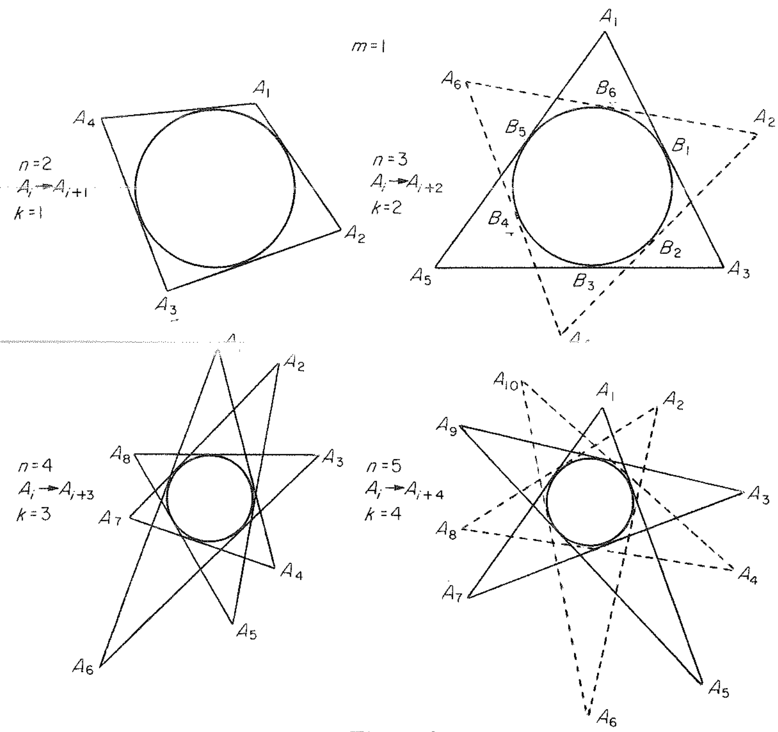

Figure 3.

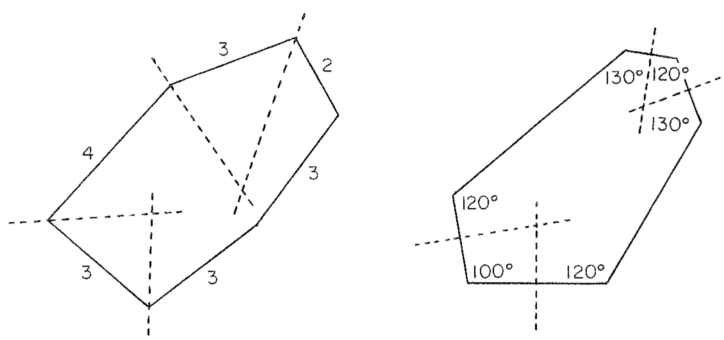

Figure. 4

Furthermore, in accordance with [4] and [6] the above generalization of Turnbull's theorem has the following interesting dual:

Theorem 2. If $A_1A_2\ldots A_{2n}$ $(n>1)$ is any circumscribed $2n$-gon in which vertex $A_i \to A_{i+k}$, then the two sums of alternate sides are equal.

In Figure 3, the first four dual examples for $m=1$ are given. Note that in theorem 1 the sum of the alternate angles is equal to half the total angle sum, while in the dual result we have the sum of the alternate sides equal to half the perimeter. However, in contrast to theorem 1, the two sums of alternate sides are not constant for the same value of m, as the perimeter of a circumscribed $2n$-gon varies, not only with respect to the size of the circle, but also for a circle with a given radius. Furthermore, it should be noted that it is necessary here to give a more general meaning to the concept 'side'. For example, for $n=3$ the 'sides' are to be interpreted respectively as $A_1B_1+A_2B_6$, etc. In other words, a 'side' is the sum of the intersecting tangents drawn from adjacent vertices. As before, the proof of theorem 2 is based on the theorem that the tangents drawn from a point ouside a circle are equal, and is left to the reader.

Unfortunately the converses are only valid for cyclic and circumscribed quadrilaterals ($n=2$, $k=1$). The first figure in Figure 4, for example, shows a convex hexagon for which the two sums of alternate sides are equal, but a circle cannot be inscribed in it (the angle bisectors are not concurrent). Similarly, the second figure in Figure 4 shows a convex hexagon with the two sums of alternate angles equal, but a circle cannot be circumscribed around it (the perpendicular bisectors of the sides are not concurrent). In the same way, counter-examples can be constructed for the other cases.

End note: [1]For $k=2, 4, 6, 8\ldots$ we can easily formulate and prove the further generalization that for non-cyclic $2n$-gons of this kind the two sums of alternate interior angles are also each equal to $m\pi$, but then it would not include the original version of Turnbull's theorem nor the other generalizations.

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