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Generalizing a problem of Sylvester

MICHAEL DE VILLIERS

"*Mathematics is not a book confined within a cover and bound between brazen clasps, whose contents it needs only patience to ransack; it is not a mine, whose treasures may take long to reduce into possession, but which fill only a limited number of veins and lodes; it is not a soil, whose fertility can be exhausted by the yield of successive harvests; it is not a continent or an ocean, whose area can be mapped out and its contour defined: it is limitless as that space which it finds too narrow for its aspirations; its possibilities are as infinite as the worlds which are forever crowding in and multiplying upon the astronomer's gaze*." - James Joseph Sylvester quoted in Stewart, I. (2010). *Hoard of Mathematical Treasures*, Profile Books: London, p. 98.

The Euler line of a triangle is mostly valued, not for any practical application, but purely as a beautiful, esoteric example of post-Greek geometry. Much to the author's surprise, however, he recently came across the following result and theorem by the British mathematician James Joseph Sylvester (1814-1897) in [1] that involves an interesting application of forces that relate to the Euler line (segment). This result is also mentioned in [2] without proof or reference to Sylvester.

Theorem of Sylvester

The resultant of three equal forces OA, OB and OC acting on the circumcentre O of a triangle ABC, is the force represented by OH, where H is the orthocentre of the triangle.

Given the remarkable analogy between the nine-point circle and Euler line on the one hand, and that of the Spieker circle and Nagel line on the other hand, as discussed in [3], [4] and [5], it was not surprising when checking with *Sketchpad* to find that the following analogous result is also true.

Nagel segment theorem

The resultant of three equal forces IA, IB and IC acting on the circumcentre I of a triangle ABC, is the force represented by IN, where N is the Nagel point of the triangle.

Reflecting on (and proving) why the result was true in both cases, led the author to the following further generalization, which includes the above two as special cases.

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Generalization

The resultant of three forces *PA*, *PB* and *PC* acting on any point *P* of a triangle *ABC*, is the force represented by *3PG*, where *G* is the centroid of the triangle.

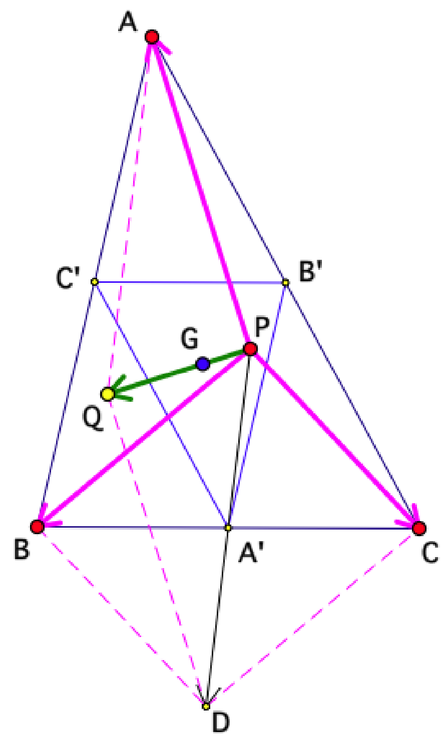

Figure 1

Proof

Consider Figure 1. By completion of the parallelogram of forces, the resultant of forces *PB* and *PC* is *PD*. Similarly, completing the parallelogram *AQDP*, we find *PQ* as the resultant of the forces *PA* and *PD*.

But it is well known that the median triangle *A'B'C'* is similar to *ABC* and that a halfturn around *G* and a dilation with scale factor 1/2 maps *ABC* onto *A'B'C'*. Now note that *A'P* = 1/2 *AQ* and that since *A'P* and *AQ* are parallel, they are equally inclined towards the respective sides of triangles *A'B'C'* and *ABC*. Hence, *P* and *Q* are in the same 'relative positions' with respect to the similar triangles *A'B'C'* and *ABC*. Thus, a halfturn around *G* and a dilation with scale factor 1/2 also maps *Q* onto *P*, and it follows that *PQ* = 3*PG*. Q.E.D.

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As is often the case, this generalization is not original as the author later found it mentioned without proof in [6] and attributed to Alison in 1885 [7]. However, what is not mentioned in [6] is, that due to the half-turn relationship between *ABC* and *A'B'C'*, the general result also applies, with only slight modifications, if the three forces acting on *P* are *P A'*, *P B'* and *P C'*, and is left to the reader to verify.

Furthermore, and perhaps more importantly, the result generalizes in exactly the same way to any quadrilateral as follows.

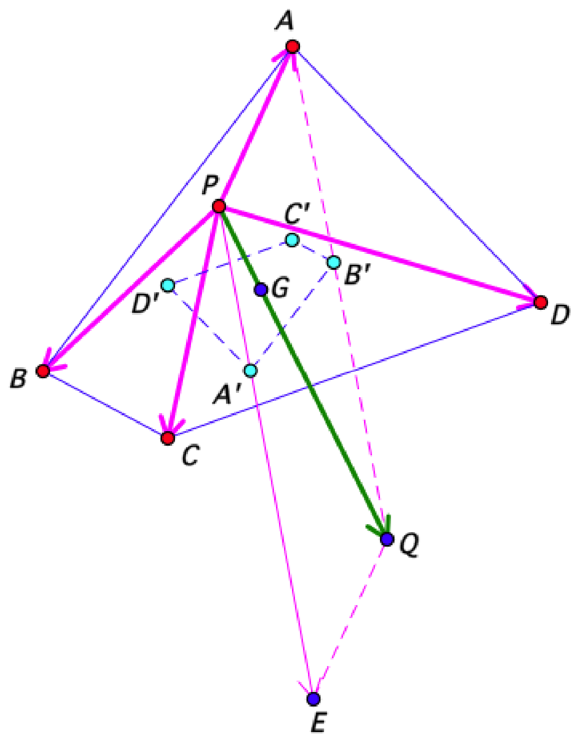

Figure 2

Quadrilateral generalization

The resultant of four forces *PA*, *PB*, *PC* and *PD* acting on any point *P* of a quadrilateral *ABCD*, is the force represented by *4PG*, where *G* is the centroid of the quadrilateral.

Proof

Consider Figure 2. In general, the centroid of any quadrilateral *ABCD* is defined as the centre of similarity *G* between *ABCD* and *A′B′C′D′* where *C′*, *D′*, *A′* and *B′* are the respective centroids of triangles *ABD*, *ABC*, *BCD* and *CDA* (and *ABCD* maps onto *A′B′C′D′* with a dilation -1/3, i.e. a halfturn and a reduction by 1/3) [8].

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According to the general theorem for a triangle above, the resultant for the three forces PB, PC and PD is $PE = 3\ PA'$. By completion of the parallelogram of forces $PAQE$ the resultant of all four forces is therefore given by PQ. But since $A'P$ is parallel and equal to 1/3 AQ, it follows that P and Q are in the same 'relative positions' with respect to the similar quadrilaterals $A'B'C'D'$ and $ABCD$. Thus, a halfturn around G and a dilation with scale factor 1/3 also maps Q onto P, and it follows that $PQ = 4PG$. Q.E.D.

Using the general, geometric theorem and definition of the centroid of any polygon from [8]: "Given a n-gon $A_1A_2A_3...A_n$ ($n \geq 3$), then the centroids of the (n-1)-gons, $A_1A_2A_3...A_{n-1}$, $A_2A_3A_4...A_n$, etc. that subdivide it, form a n-gon $A_1^{'}A_2^{'}A_3^{'}...A_n^{'}$ similar to the original n-gon with a scale factor of $-\dfrac{1}{n-1}$, while the centre of similarity is the centroid of the original n-gon", it's now easy to see that the following interesting result is true for any polygon (and can be proved by mathematical induction).

Polygon generalization

The resultant of n forces $PA_1, PA_2, ..., PA_n$ acting on any point P of a polygon $A_1A_2A_3...A_n$ ($n \geq 3$) is the force represented by nPG, where G is the centroid of the quadrilateral.

Further reflections

1) A few days after submission of the above paper, the author suddenly thought again of the general result while driving to practice tennis, and found the following trivial proof while driving. If we use coordinates (or vectors), placing the origin at the point P, then the resultant of all the forces is the sum of all the x and y coordinates of the forces respectively, and by definition of the centre of gravity, G is located at the point (sum of x-coordinates)/n, (sum of y-coordinates)/n, from which the result immediately follows! More over, it is immediately clear that the same result would hold in space by the same argument!

2) Somewhat later, the author managed to find a copy of Alison's original paper [7], and perhaps not surprisingly, after proving the general case for a triangle somewhat differently, Alison extends it to the case for a tetrahedron, and then generalizes it further

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to n points in space. So the interesting generalization above is not new at all, but unfortunately seems not to be well known.

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MICHAEL DE VILLIERS

School of Science, Mathematics & Technology Education,

University of KwaZulu-Natal, South Africa

e-mail: *profmd@mweb.co.za*

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