

# Simply Symmetric<sup>1</sup>

MICHAEL DE VILLIERS

*University of KwaZulu-Natal, South Africa*

e-mail: [profmd@mweb.co.za](mailto:profmd@mweb.co.za)

<http://mysite.mweb.co.za/residents/profmd/homepage4.html>

“ ... *symmetric means something like well-proportioned, well-balanced, and symmetry denotes that sort of concordance of several parts by which they integrate into a whole. Beauty is bound up with symmetry.*”  
- Hermann Weyl (1989, p. 3)

Symmetry is found in the visual arts, architecture and design of artefacts since the earliest time. Many natural objects, both organic and inorganic, display symmetry: from microscopic crystals and sub-atomic particles to macro-cosmic galaxies. Today it features strongly in higher mathematics such as Linear and Abstract Algebra, Projective and Fractal Geometry, Algebraic Topology, Graph and Function Theory, etc., and in many other mathematical disciplines such as Quantum Physics, Relativity, String Theory, etc.

Most primary geometry curricula around the world introduce the concept of line symmetry fairly early, and sometimes also that of rotational, translational and glide reflective symmetry. However, in high school and even college geometry textbooks, one still tends to find a general lack of using symmetry to not only *organize* and *classify* geometric objects, but also to more easily *logically explain* (prove) certain properties of these objects as theorems.

For example, typically concepts like rectangles and rhombi are respectively introduced by definitions such as: “*A rectangle is a quadrilateral with all angles equal*” and “*A rhombus is a quadrilateral with all sides equal.*”

Apart from the problematic pedagogic<sup>2</sup> and epistemological<sup>3</sup> issues that students are just provided ‘*ready-made*’ definitions from the start that they must dutifully

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<sup>2</sup> According to the Van Hiele theory of learning geometry, formal definitions of concepts should not be introduced before students have reached Van Hiele Level 3.

memorize, these definitions do not express the line symmetric properties of these objects at all. More-over, many (if not most) textbooks such as Alexander & Koeberlein (2007, p. 212) provide a summative list of the properties of the quadrilaterals, which do not even mention a single symmetric property, despite an earlier chapter dealing generally with line and point symmetry!

It is sad that students are generally not given the opportunity to critically discuss and compare different possible definitions of mathematical objects such as the quadrilaterals. Such an exercise could do much to dispel the common misconception among students that definitions are somehow cast in stone, and that there is only one (correct) definition for each defined object in mathematics. It can help students see that mathematical definitions are not independent Platonic objects, existing unchallengeable in some ideal world, but are open to critique, and very often are a matter of choice and convenience. As Freudenthal (1973:418) has argued, it can help students learn about organizing and conceptualizing a topic not only in different ways, but more importantly, develop some understanding of "**why** *some organisation, some concept, some definition is better than another*" (bold added).

For example, by defining a rectangle as "*a quadrilateral with two lines of symmetry, each through a pair of opposite sides*" is not only elegant, but allows one to much more easily derive (prove) the other properties of a rectangle, without having to use congruency. For instance, consider Figure 1. By reflection around the line of symmetry  $m$ , angles  $BAD$  and  $ABC$  respectively map onto angles  $CDA$  and  $DCB$ ; hence  $\angle BAD = \angle CDA$  and  $\angle ABC = \angle DCB$ . By now reflecting around the line of symmetry  $n$ , we similarly find  $\angle BAD = \angle ABC$  and  $\angle CDA = \angle DCB$ ; hence all angles are equal.

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<sup>3</sup> Research mathematics usually starts with problems and seldom with definitions of concepts, and only end with definitions.

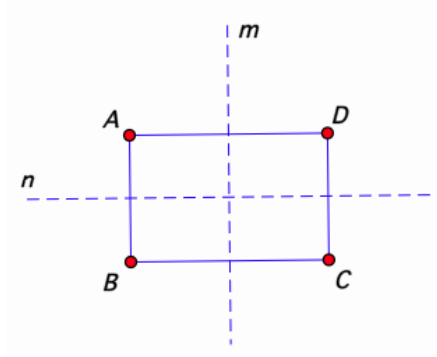


Figure 1

To explain why (prove) its diagonals are equal is also simple and straightforward, with absolutely no need to use any congruency. For example, line segment  $AC$  maps onto line segment  $DB$  under the reflection around line  $m$ ; hence  $AC = DB$ .

If we now further view and classify a rectangle as a special parallelogram, there is obviously no need to prove that it has opposite sides equal and parallel, because it then *inherits* those properties from the parallelogram. However, if need be, one can also easily derive these properties of a rectangle from its symmetry definition above. For example, it's easy to see by reflection that opposite sides map onto each other, and from the fact that all angles are equal, it follows that co-interior angles are supplementary, and therefore opposite sides are parallel.

Similarly we can more easily, and elegantly derive and explain the properties of a rhombus without using any congruency by simply defining it as “a quadrilateral with two axes of symmetry, each through a pair of opposite vertices.” Again by reflecting around the axes of symmetry, we easily find all sides equal, diagonals bisecting each pair of opposite angles, and perpendicular to each other (from the definition of a reflection).

The same applies to a parallelogram, which can be defined as “a quadrilateral with half-turn symmetry.” From this definition, and the definition of a half-turn as a rotation through  $180^\circ$ , it follows immediately, with absolutely no need for congruency proofs that it has opposite sides equal and parallel, opposite angles equal, and diagonals bisecting each other.

A similar ‘deductive economy’ is achieved if we define an isosceles trapezium as “a quadrilateral with at least one axis of symmetry through a pair of opposite sides” and

a kite as “a quadrilateral with at least one axis of symmetry through a pair of opposite vertices”.

Symmetry is also extremely valuable and useful in problem solving and proving other theorems and results. For example, the author recently found the following interesting little problem from Posamentier & Salkind (1996, p. 4):

*“The trisectors of the angles of a rectangle are drawn. For each pair of adjacent angles, those trisectors that are closest to the enclosed side are extended until a point of intersection is established. The line segments connecting those points of intersection form a quadrilateral. Prove that the quadrilateral is a rhombus.”*

In the provided solutions by the authors on p. 60, a long series of congruency arguments are provided to establish the result (and which are not repeated here for the sake of expediency). However, the result follows almost immediately from symmetry and to use congruency arguments here are really not elegant or economical at all.

Consider Figure 2 where  $ABCD$  is the rectangle with the trisectors constructed to form quadrilateral  $EFGH$ . Since  $ABCD$  is symmetrical and the constructions are symmetrically carried out, it follows that the reflection of  $E$  around the line  $m$  will be  $G$ , and vice versa. Similarly,  $H$  maps to  $F$  under a reflection around line  $n$ . Therefore,  $EFGH$  has two axes of symmetry through opposite pairs of vertices, and hence is a rhombus.

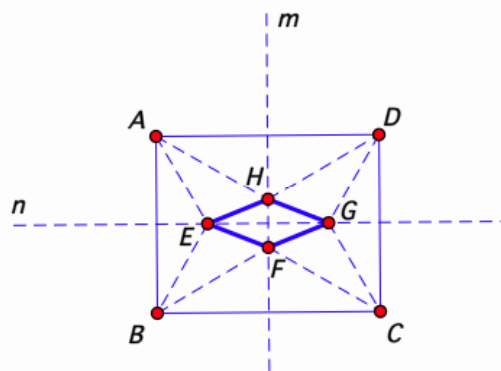


Figure 2

Another alternative, but equivalent logical explanation would be to argue that  $H$  and  $F$  both lie on  $m$  since they are at the apexes of isosceles triangles, etc. It seems somewhat silly and pedantic to insist here that students be shackled by the traditional Euclidean approach of only using congruency in proof. Why not allow students the flexibility and freedom to use powerful and elegant symmetry arguments such as these?

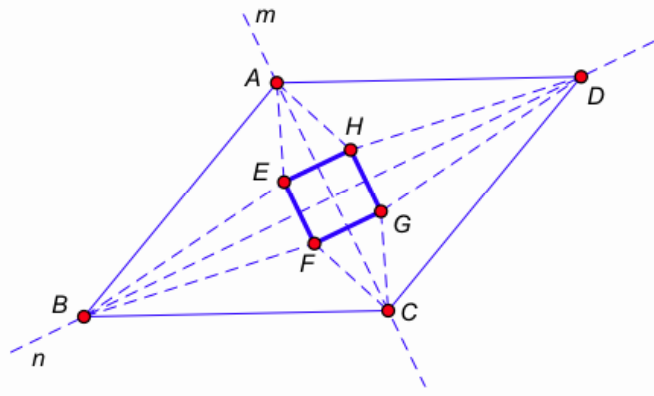


Figure 3

More-over from symmetry, it is now easy to see that if the same construction is carried out for a rhombus, we would similarly obtain a rectangle as shown in Figure 3. Here the axes of symmetry  $m$  and  $n$  of the rhombus  $ABCD$  are clearly also axes of symmetry of  $EFGH$ , but through its sides, and hence the result.

If the same symmetrical trisector construction is carried out on a parallelogram  $ABCD$  as shown in Figure 4, we immediately have that the constructed figure  $EFGH$  will also have half-symmetry, and will therefore be a parallelogram. Once again, there's no need for lengthy congruency arguments!

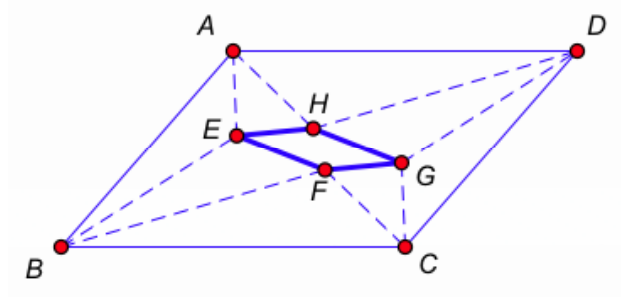


Figure 4

From symmetry too it follows that if the same trisector constructions are respectively carried out on an isosceles trapezium and kite, we would respectively obtain a kite and an isosceles trapezium as shown in Figure 5.

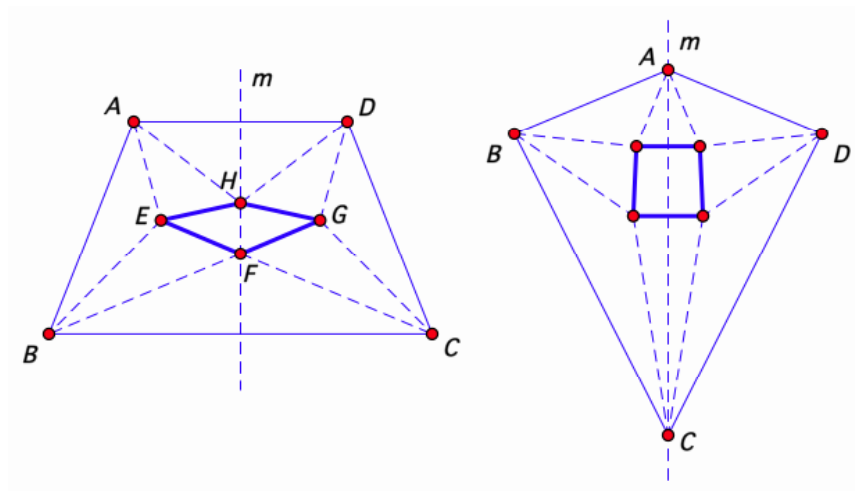


Figure 5

Given the inherit symmetry of the quadrilaterals, and of the construction itself, the above results automatically generalize for quadri-sectors, penta-sectors, etc. of the angles. Figure 6 shows the respective hexa-sector constructions for a rectangle and a rhombus.

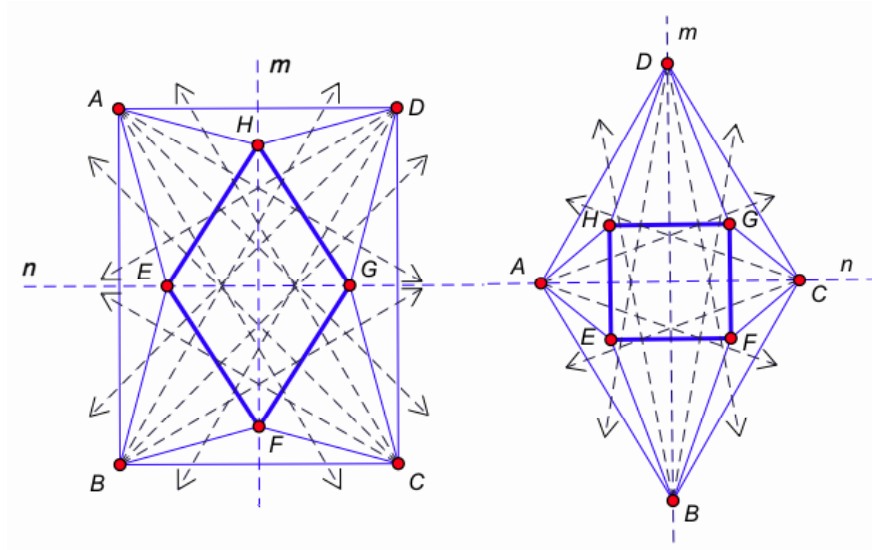


Figure 6

It is hoped that these few examples have convinced some readers of the power and beauty of symmetry, and to encourage its use more often in high school geometry where appropriate (also compare Fielker, 1983; Johnston-Wilder & Mason, 2005). Note that it is not suggested here that we should throw out the Euclidean approach completely, but really more that we should have some balance and flexibility, and sometimes allow (accept) symmetry arguments where it is sensible to do so. For example, in some

instances, as illustrated here, we are perhaps needlessly be-labouring the point when a simple argument in terms of symmetry will suffice (and in fact, might be more *illuminating*).

After all, in mathematical papers that mathematicians publish they frequently use symmetry arguments in geometry (and in other areas) rather than always resorting to tedious congruency proofs, etc. The same applies to the many mathematics competitions where students are usually not penalized if they instead use symmetry (and in fact are sometimes given extra marks for elegance to distinguish between top candidates). It seems unreasonable, and perhaps even unfair, to insist that learners at school should struggle the 'hard' way.

It should, however, not be an all or nothing approach. Learners are likely to benefit if concepts are sometimes approached and seen, and theorems proved, from both perspectives, Euclidean as well as a more modern transformational view, as this is likely to increase their understanding.

Lastly, it is hoped that as discussed more fully in De Villiers, Govender & Patterson (2009), students will be more actively engaged not only in the process of formulating and evaluating different definitions for the quadrilaterals, including ones based on their underlying symmetry, but also to critically compare them in terms of elegance and deductive economy. For example, comparing different definitions for say a parallelogram or a kite might help them understand why one definition might be better than another, and here we are also referring to the whole class-inclusion issue of special cases as well.

*Note:* Interactive Java sketches for the results in Figures 2 and 3 are available online at:

<http://math.kennesaw.edu/~mdevilli/simply-symmetric.html>

<http://math.kennesaw.edu/~mdevilli/simply-symmetric2.html>

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