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A question of balance: an application of centroids

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The following surprising geometry result provides a nice challenge to high school or undergraduate students. Given a quadrilateral *ABCD* with equilateral triangles *ABP*, *BCQ*, *CDR* and *DAS* constructed on the sides so that say the first and third are exterior to the quadrilateral, while the second and the fourth are interior to the quadrilateral, prove that quadrilateral *PQRS* is a parallelogram.

This result appears to be reasonably well-known and can be found as an exercise in [1], and the special case when two of the vertices of *ABCD* coincide to form a triangle, apparently also appears in some high school textbooks in Korea and other Eastern countries [2]. Less well known appears to be the following generalisation and proof of the result in [3].

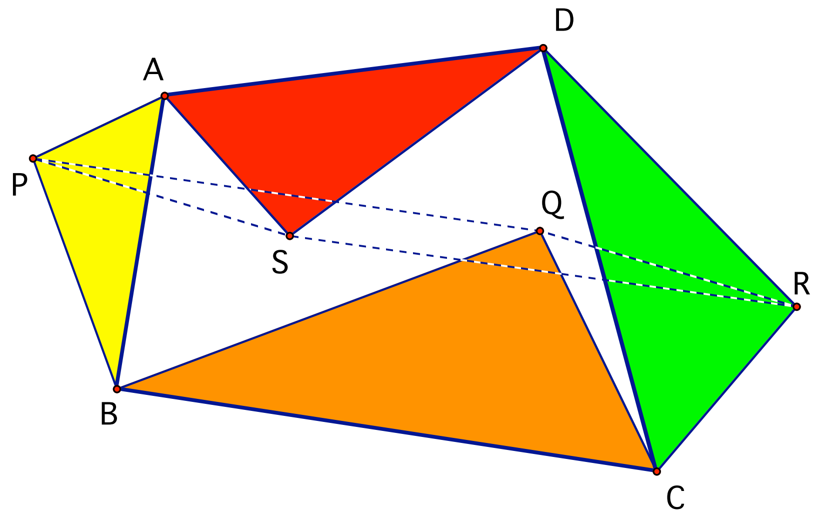

Figure 1

Theorem 1

Given a quadrilateral *ABCD* with similar triangles *PBA*, *QBC*, *RDC* and *SDA* constructed on the sides so that say the first and third are exterior to the quadrilateral, while the second and the fourth are interior to the quadrilateral, then quadrilateral *PQRS* is a parallelogram (see Figure 1).

The following interesting Van Aubel like extension of Theorem 1 was recently found with the aid of *Sketchpad*. I'm also grateful to John Silvester, King's College London for his kind assistance in this revised version.

Theorem 2

Given four points A, B, C, D, and four directly similar quadrilaterals AP_1P_2B, CQ_1Q_2B, CR_1R_2D, AS_1S_2D with respective centroids P, Q, R, S, let K, L, M and N be the midpoints of the segments P_1Q_2, Q_1R_2, R_1S_2 and S_1P_2 respectively (or of P_1S_2, S_1R_2, R_1Q_2 and Q_1P_2), and let V, W, X be the centroids of the quadrilaterals $ABCD$, $PQRS$, $KLMN$ respectively (see Figure 2). Then:

(i) $PQRS$ is a parallelogram;

(ii) $KLMN$ is a parallelogram; and

(iii) W is the midpoint of the segment VX.

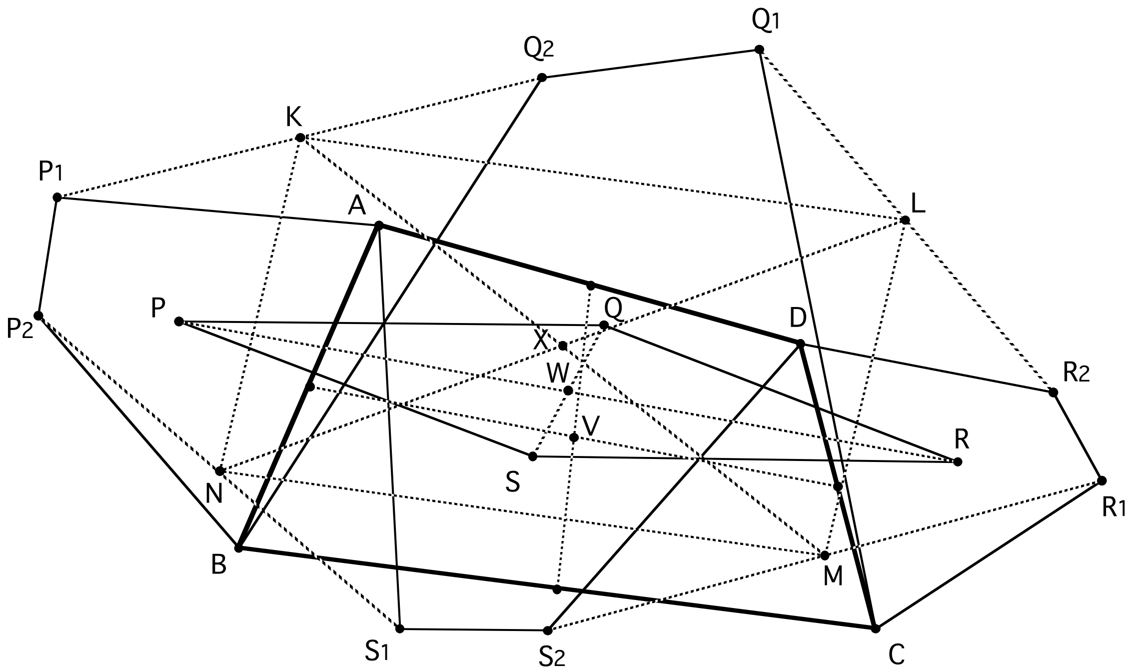

Figure 2

Let us first prove the following Lemma that will be useful in proving this result.

Lemma

Given two parallelograms $ABCD$ and $IJKL$, the midpoints E, F, G, H of the segments AI, BJ, CK, DL form another parallelogram (see Figure 3). (The diagram shows a case with the parallelograms both labelled anticlockwise, but both lemma and proof work equally well in all cases.)

Proof

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Writing $\mathbf{a} = (a_1, a_2)$ for the vector representing *A*, etc, the condition for *ABCD* to be a parallelogram is $\mathbf{a} - \mathbf{b} = \mathbf{d} - \mathbf{c}$ (opposite sides, equal length and parallel) or equivalently $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$. Similarly, for *IJKL*, we have $\mathbf{i} + \mathbf{k} = \mathbf{j} + \mathbf{l}$, from which follows $(\mathbf{a} + \mathbf{i}) + (\mathbf{c} + \mathbf{k}) = (\mathbf{b} + \mathbf{j}) + (\mathbf{d} + \mathbf{l})$. Dividing by 2, we have the condition for the four midpoints to form a parallelogram (provided they are not collinear, or coincident, which they might be; but we'll regard this as a degenerate parallelogram). In addition: the centre *X* of *ABCD* is $(\mathbf{a} + \mathbf{c})/2$, the centre *Z* of *IJKL* is $(\mathbf{i} + \mathbf{k})/2$ and the centre *Y* of *EFGH* is $[(\mathbf{a} + \mathbf{i}) + (\mathbf{c} + \mathbf{k})]/4$, which shows that *Y* is the midpoint of *XZ*.

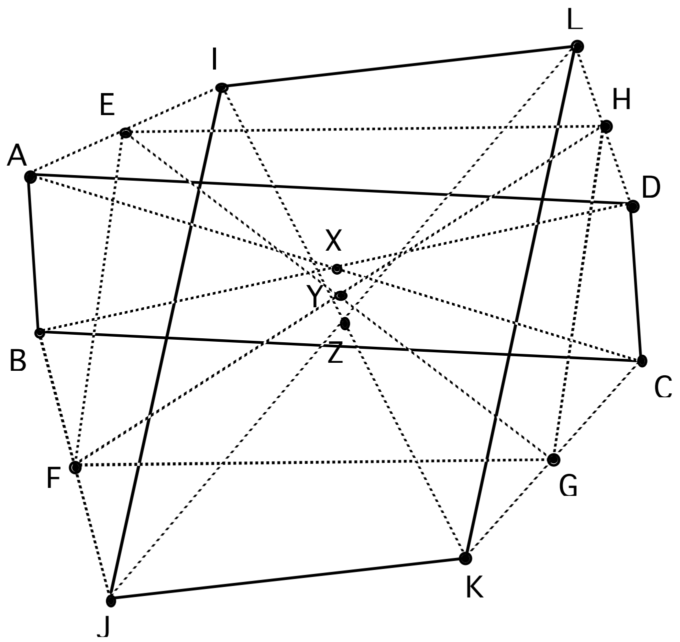

Figure 3

Proof of Theorem 2

By Theorem 1, $P_1Q_1R_1S_1$ and $P_2Q_2R_2S_2$ are both parallelograms, and therefore by the above lemma, *KLMN* is also a parallelogram.

Assume a mass of 2 is placed at each of *A*, *B*, *C*, *D*, and a unit mass at the other eight places: 16 altogether. Then for the quadrilateral AP_1P_2B we take the unit masses at P_1 and P_2, and one each of the 2 at *A* and *B*; the other mass at *A* gets used up in AS_1S_2D and the other one at *B* in CQ_1Q_2B. Doing the same for the other quadrilaterals it follows that *P*, *Q*, *R* and *S* are the respective centroids of the masses placed at the vertices of the four quadrilaterals surrounding them. Thus, *W* the centroid of parallelogram *PQRS* is the centroid of all sixteen masses.

Note that the centroid of a quadrilateral is the same as the Varignon centre, the centre (or centroid) of the parallelogram formed by the midpoints of the sides of the quadrilateral, namely V. If we now only consider the masses at the eight outer vertices, then their centroid is located at X, the centroid of the parallelogram $KLMN$. When we now collect up, we have $2 + 2 + 2 + 2 = 8$ at V and the other 8 at X, so altogether W is the balancing midpoint as required.

Notes

1) Dynamic Geometry (*Sketchpad 4*) sketches in zipped format (Winzip) of the results discussed here can be downloaded directly from:
http://mysite.mweb.co.za/residents/profmd/balance.zip
(If not in possession of a copy of *Sketchpad 4*, these sketches can be viewed with a free demo version of *Sketchpad 4* that can be downloaded from:
http://www.keypress.com/sketchpad/sketchdemo.html)

2) Or simply view and manipulate a JavaSketchpad version online at:
http://math.kennesaw.edu/~mdevilli/balance.html

References

1. I.M. Yaglom, *Geometric Transformations I*. Washington, DC: Mathematical Association of America, p. 39, (1962).
2. H. Lew, Pappus in Modern Dynamic Geometry: An Honest Way for Deductive Proof. Paper to be presented at ICMI Study Conference 17, *Digital technologies and mathematics teaching & learning*, Hanoi Institute of Technology, 3-8 December 2006.
3. M. de Villiers, The Role of Proof in Investigative, Computer-based Geometry: Some personal reflections. Chapter in D. Schattschneider & J. King, *Geometry Turned On!* Washington: MAA, pp. 15-24, (1997).

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