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Relations between the sides and diagonals of a set of hexagons

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"*The art is not in the "truth" but in the explanation, the argument. It is the argument itself which gives the truth its context, and determines what is really being said and meant. Mathematics is the* ***art of explanation****.*" – Paul Lockhart [1]

An interesting parallelogram theorem states that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of its diagonals. For a parallelogram *ABCD*, since opposite sides are equal, we therefore have that $2(AB^2 + BC^2) = AC^2 + BD^2$. It is most easily proved with the cosine rule or using vectors. The result is also called the "Parallelogram Law" on Wikipedia [2] and some other sites, but this name may be easily confused with the perhaps better-known parallelogram law of forces in physics.

Recently the author was considering how this theorem might be generalized to a *parallelo-hexagon*, in other words, what relationships, if any, could be found between the sum of the squares of the sides of *a hexagon with opposite sides equal and parallel*, and the sums of the squares of its major diagonals (which connect opposite vertices). Using *Sketchpad*, the author experimentally found the following interesting inequality for a parallelo-hexagon *ABCDEF* (see Figure 1): $AD^2 + BE^2 + CF^2 \leq 4(AB^2 + BC^2 + CD^2)$. The reader is now invited to explore the result dynamically at the URL http://dynamicmathematicslearning.com/parm-law-hexagon.html before continuing.

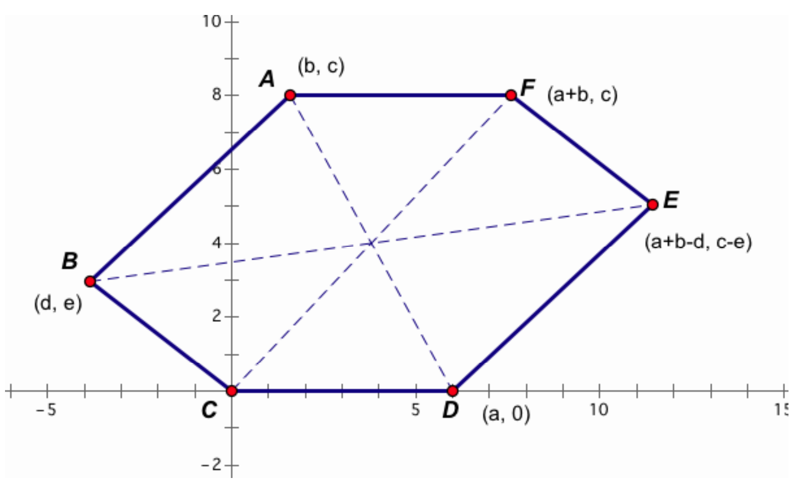

Figure 1

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From experimental exploration up to that point, the author thought the equality was only true for a regular hexagon, and is easy to prove, since a regular hexagon has all sides equal and diagonals equal, and $AB = AD/2$, from which the result follows. However, as we shall see later from the general proof, under certain conditions, the equality holds for a more general type of parallelo-hexagon. Further exploration also revealed that the inequality was valid even for a concave or crossed parallelo-hexagon. But *why* was the result true? The experimental conviction obtained from dragging provided no explanation at all: all it gave was more and more empirical evidence (compare [3]).

After trying unsuccessfully for a while to deduce the general result by applying the parallelogram theorem to the three different parallelograms that make up a parallelo-hexagon, the author decided to switch to a coordinate geometry approach.

Proof

Consider Figure 1 where a parallelo-hexagon is placed on a coordinate grid with vertices C, D, A and B having respective coordinates $(0, 0)$; $(a, 0)$; (b, c) and (d, e). It then follows from the symmetric properties of a parallelo-hexagon that the respective coordinates of vertices E and F are $(a + b - d, c - e)$ and $(a + b, c)$. This gives us the following two equations:

$$4(AB^2+BC^2+CD^2)=4\left[(c-e)^2+(b-d)^2+d^2+e^2+a^2\right]$$

$$AD^2+BE^2+CF^2=c^2+(b-a)^2+(a+b-2d)^2+(c-2e)^2+(a+b)^2+c^2$$

Expanding and subtracting the second equation from the first, gives us:

$$\begin{aligned}4(AB^2+BC^2+CD^2)-(AD^2+BE^2+CF^2)&=a^2+b^2+c^2+4d^2+4e^2-2ab+4ad-4bd-4ce\\&=(a-b)^2+4d(a-b)+4d^2+(c-2e)^2\\&=(a-b+2d)^2+(c-2e)^2,\end{aligned}$$

which completes the proof, since the difference of these equations is the sum of two squares, which is always greater and equal to zero.

Looking back

From this equation, note that equality will only hold when $a - b + 2d = 0$ and $c = 2e$; hence, the parallelo-hexagon need not be regular as originally thought. This is therefore an illustrative example of what is called the *discovery* function of proof in [4], whereby proving a result might lead to further unexpected discoveries.

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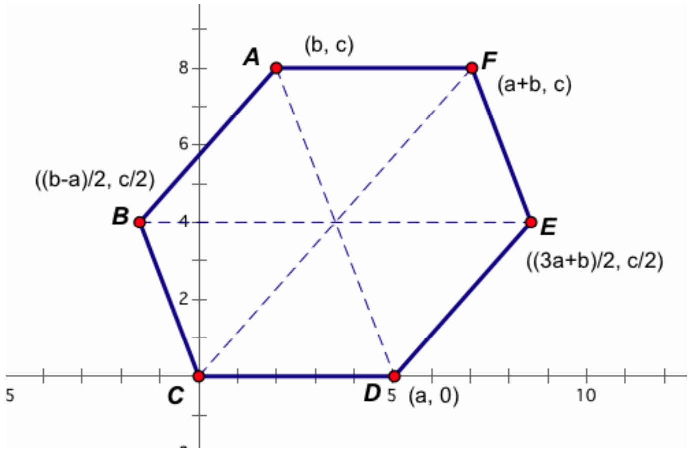

Figure 2

More-over, we can easily deduce from these conditions for equality by substituting $d = (b-a)/2$ and $e = c/2$ as shown in Figure 2, that a parallelo-hexagon has the property $AD^2 + BE^2 + CF^2 = 4(AB^2 + BC^2 + CD^2)$, if and only if, its diagonals are parallel to a pair of opposite sides! For example, it's easy to see in Figure 2 that BE has the same zero slope as AF and CD, and that AB, FC and ED all have the same slope $c/(a + b)$, and that BC, AD and FE all have the same slope $c/(b - a)$. Conversely, from Figure 1, if we have $BE \,//\, AF$, then $e = c - e$ which gives us the one condition $c = 2e$. And if AD is also parallel to BC in Figure 1, then $c/(b - e) = e/d$ which simplifies to the other condition $d = (b - a)/2$.

A simpler, alternative proof

The author is grateful to the anonymous referee for pointing out that if we chose the centre of symmetry of the parallelo-hexagon as origin, and A, B and C respectively as (a, d), (b, e) and (c, f) with D, E and F then respectively becoming $(-a, -d)$, $(-b, -e)$ and $(-c, -f)$, the result reduces algebraically more easily to a sum of two squares, e.g. $4\left((a-b+c)^2 + (d-e+f)^2\right)$. This shows how critical the choice of origin can sometimes be.

Another generalization

Of related interest is that this parallelogram theorem generalizes to any quadrilateral $ABCD$, and states that $AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2 + 4x^2$ where x is the distance between the midpoints of the diagonals [5]. Douglas [6] goes even further and generalizes the result to any $2n$-gon. The special case of the Douglas theorem for a parallelo-hexagon is shown in Figure 3, for which the following holds:

$2(AB^2+BC^2+CD^2)-2(AC^2+CE^2+EA^2)+AD^2+BE^2+CF^2=9MN^2$, with M and N the respective centroids of triangles ACE and BDF.

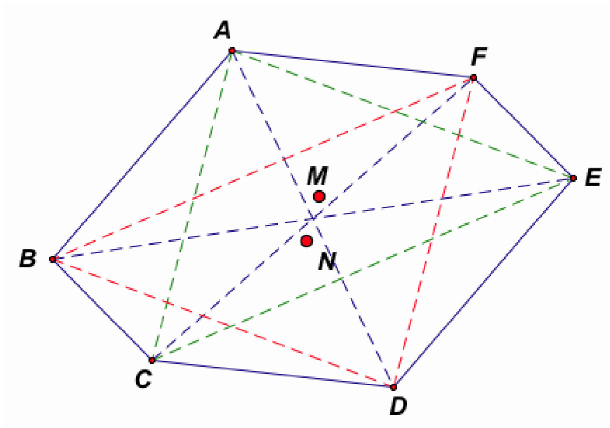

Figure 3

If we have a parallelo-hexagon with the property as shown in Figure 2 with diagonals parallel to opposite sides, then it's easy to see that M and N will coincide. By substituting the previous equality into Douglas' one above, the following two relationships will then hold for this special type of parallelo-hexagon:

$3(AB^2+BC^2+CD^2)=(AC^2+CE^2+EA^2)$ and $3(AD^2+BE^2+CF^2)=4(AC^2+CE^2+EA^2)$.

Further experimental exploration revealed that for a general parallelo-hexagon the following equality holds: $(AB^2+BC^2+CD^2)+(AC^2+CE^2+EA^2)=AD^2+BE^2+CF^2$. This result can be easily proved using the coordinates in Figure 1 and is left to the reader check.

Rewriting this equality as:

$(AB^2+BC^2+CD^2)-(AD^2+BE^2+CF^2)=-(AC^2+CE^2+EA^2)$ and substituting it in Douglas' equality, we obtain: $4(AB^2+BC^2+CD^2)-(AD^2+BE^2+CF^2)=9MN^2$.

Since the right hand side of this equality for a parallelo-hexagon is always greater and equal to zero, we again obtain our inequality of earlier, with equality holding, when M and N coincide (giving us a parallelo-hexagon with diagonals respectively parallel to a pair of opposite sides).

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A conjecture and its counter-example

While initially investigating a possible relationship between the sum of the squares of the sides of a parallelo-hexagon and the sum of the squares of its main diagonals, the author made the following lower bound conjecture for a convex parallelo-hexagon: $3(AB^2+BC^2+CD^2) < AD^2+BE^2+CF^2$.

The result is valid if all the angles of the parallelo-hexagon are obtuse and can be proved as follows. Applying the parallelogram law to parallelogram *ABDE* in Figure 1, we obtain $2(AB^2+BD^2) = AD^2+BE^2$. But if angle *BCD* in triangle *BCD* is obtuse, it follows that $BD^2 > BC^2+CD^2$. Therefore, $2(AB^2+BC^2+CD^2) < AD^2+BE^2$. Similarly, if angles *ABC* and *BAF* with respect to parallelograms *ACDF* and *BCEF* are obtuse, it follows that $2(CD^2+AB^2+BC^2) < AD^2+CF^2$ and $2(AB^2+BC^2+CD^2) < BE^2+CF^2$. By adding these three equations and simplifying, the lower bound follows: $3(AB^2+BC^2+CD^2) < AD^2+BE^2+CF^2$.

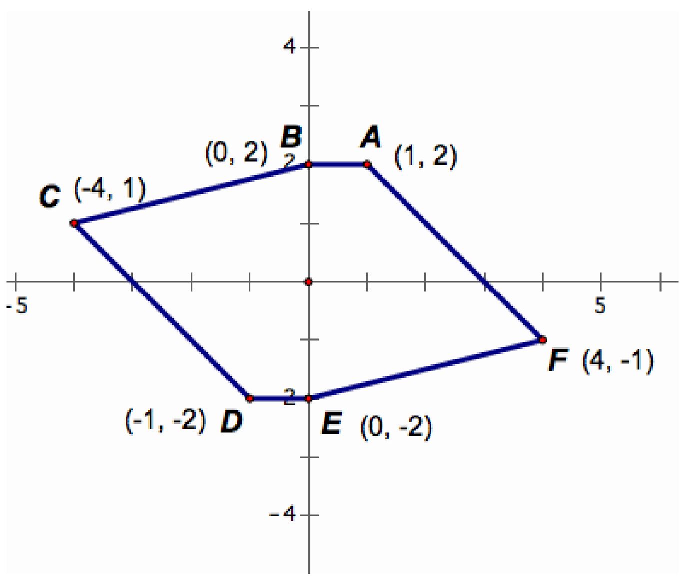

Figure 4

However, this conjecture is not generally true for convex parallelo-hexagons. The author is grateful to the referee for giving the following simple counter-example as shown in Figure 4 for a symmetrically placed hexagon. Let *A*, *B* and *C* be (1,2), (0,2), (-4, 1), etc. The hexagon is convex, but as is left to verify by the reader $3(AB^2+BC^2+CD^2) = 108 > AD^2+BE^2+CF^2 = 104$. (The dynamic sketch at the URL

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given earlier can also be used to produce a counter-example, e.g. by dragging A to shorten AF and dragging E to make opposite angles B and E more acute.)

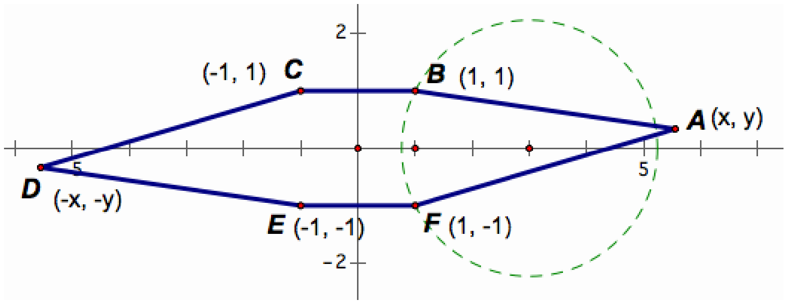

Figure 5

The referee also points out that we can generate families of convex parallelo-hexagons for which the inequality fails. For one such family take B, C as (1, 1), (-1, 1), etc., and A, D as (x, y), $(-x, -y)$, satisfying $x > 3 + \sqrt{5}$, $-1 < y < 1$ (see Figure 5). In fact, there is always a critical circle that depends on B, C and their images E, F. If A lies inside this circle the inequality holds. On the circumference it becomes an equality, outside the circle the opposite inequality is true. In the example in Figure 5, the centre of the circle is at (3, 0) and has radius $\sqrt{5}$.

Concluding comments

Borwein [7] and others have remarked that it is necessary in the light of modern computing technology to re-evaluate the role of proof, as well as the teaching of it. As we have seen in this investigation, the conviction obtained from the dynamic geometry exploration motivated the search for a proof, and had little to do with the removal of doubt. However, proving the result gave useful insight leading to further discoveries.

Since this exploration should be easily accessible to high school learners, it could be adapted into a worksheet to illustrate the so-called 'discovery' function of proof, as well as the value of some algebraic factorization and simplification by hand.

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