92.22 Maths bite: averaging polygons

In [1], David Wells mentions the following pretty result:

'Take any hexagon, and find the centres of gravity of each set of three consecutive vertices. These immediately form a hexagon whose opposite sides are equal and parallel in pairs.'

What he does not explicitly mention is that this is the second in a chain of similar results, the first being the familiar observation that the midpoints of the sides of any quadrilateral form the vertices of a parallelogram. Retaining Wells' description, the general result is:

'Take any 2n-sided polygon, and find the centres of gravity of each set of n consecutive vertices. These form a 2n-sided polygon whose opposite sides are equal and parallel in pairs.'

The proof is a 'one-liner'. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{2n}$ be the position vectors of consecutive vertices (A_k) of the $2n$-sided polygon and continue the labelling cyclically so that $\mathbf{a}_{2n+i} = \mathbf{a}_i$ for $1 \leqslant i \leqslant n-1$. The position vectors of the vertices (B_k) of the derived $2n$-sided polygon are then given by $\mathbf{b}_k = \frac{1}{n}\left(\mathbf{a}_k + \mathbf{a}_{k+1} + \ldots + \mathbf{a}_{k+n-1}\right)$ for $1 \leqslant k \leqslant 2n$. It follows that $\mathbf{b}_{k+1} - \mathbf{b}_k = \frac{1}{n}\left(\mathbf{a}_{k+n} - \mathbf{a}_k\right)$ and, replacing k by $n+k$, that $\mathbf{b}_{n+k+1} - \mathbf{b}_{n+k} = \frac{1}{n}\left(\mathbf{a}_{2n+k} - \mathbf{a}_{n+k}\right) = \frac{1}{n}\left(\mathbf{a}_k - \mathbf{a}_{n+k}\right)$. Thus the opposite sides B_kB_{k+1} and $B_{n+k}B_{n+k+1}$ are equal and parallel; indeed, they are parallel to the diagonal A_kA_{k+n} of the original $2n$-sided polygon.

Finally, it is worth noting that (A_k), (B_k) and the vertices of each parallelogram $B_kB_{k+1}B_{n+k}B_{n+k+1}$ all share a common centre of gravity since $\sum \mathbf{a}_k = \sum \mathbf{b}_k$ and $\frac{1}{4}\left(\mathbf{b}_k + \mathbf{b}_{k+1} + \mathbf{b}_{n+k} + \mathbf{b}_{n+k+1}\right) = \frac{1}{2n}\sum \mathbf{a}_k$.

Reference

1. D. Wells, *The Penguin dictionary of curious and interesting geometry*, (1991), p. 53.

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