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Solving a locus problem via generalization

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"*It is quite natural to consider specialization as a powerful problem solving strategy: one hopes that an insight gained by looking at a special case will be helpful in solving the problem in general, or that some technique which conquers a special case can be transferred to the general situation. But it may seem odd to consider generalization - the opposite of specialization - as a useful problem solving strategy, too. It turns out, however, that many particular problems are easier to solve when cast in a more general form*." - Wolfgang Schwarz (2005)

During my geometry classes I consistently try to make conjecturing a regular feature and encourage my students to come up with their own. I also regularly make a point of showing students how I go about creating and solving new problems myself. Activities like these certainly seem to help informing students about how new mathematical knowledge is created and discovered. For example, examining any given problem can inspire many new problems simply by trying out numerous changes that could lead to new investigations. One such possible change is to consider a *generalization* of the problem.

This contrasts with the problem solving strategy often emphasised in mathematics education at various levels, namely, to consider special cases of a problem. Not only are the special cases usually more easy to solve, but often allows one to identify a pattern or give some clue towards a general solution or proof. Less frequently utilised appears to be the opposite problem solving strategy, namely, to consider a more general case than the given problem. Contrary to what one might expect, the general case is sometimes easier (or at least just as easy) to solve than the special case as Polya (1954) discusses with several examples. Other examples from high school to undergraduate level mathematics are discussed in De Villiers & Garner (2008).

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The purpose of this article is to illustrate this technique with the following problem from Klamkin (1988, p.5), which should be easily accessible to undergraduate students. It came to my attention via Nunokawa (2004).

"If *A* and *B* are fixed points on a given circle and *XY* is a variable diameter of the same circle, determine the locus of the point of intersection of lines *AX* and *BY*. You may assume that *AB* is not a diameter."

Instead of going straight ahead trying to solve the problem directly, I first asked myself whether the problem couldn't be generalised to any move-able chord *XY* of *fixed length*. Quickly checking by construction on *Sketchpad*, the following generalized conjecture was immediately confirmed: "If *A* and *B* are fixed points on a given circle and *XY* is a moveable chord of fixed length of the same circle, then the locus of the point of intersection of lines *AX* and *BY* is a circle. (It's assumed here that *AB* and *XY* are not equal in length, in which case *AX* and *BY* will be parallel and only meet at infinity)."

In proving this generalization, I came up with the following proof using similarity, and which is distinctly different from those in Klamkin (1988, p. 50) and Nunokawa (2004). Though not claiming that it is "easier" or "simpler" than the original proofs, I personally found it more explanatory of why the result is true in the special as well as the general case.

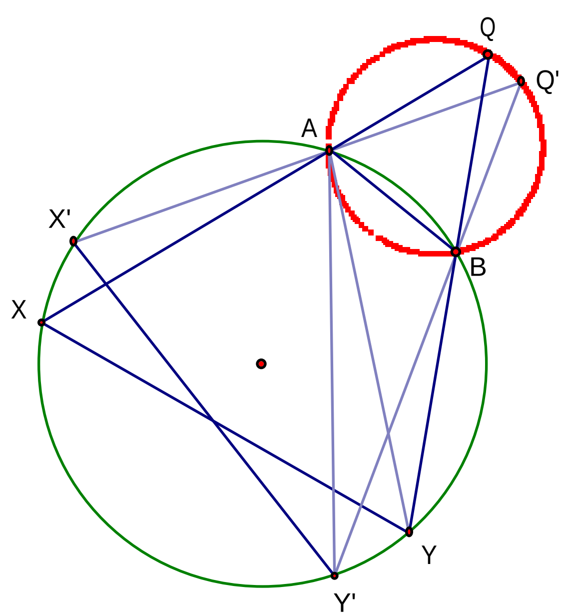

Figure 1

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Proof

Given chord *XY* of fixed length, rotate it along the circle to an arbitrary position *X'Y'* as shown in Figure 1. Since chords *YY'* and *XX'* are equal, $\angle YAY' = \angle XAX'$. But $\angle XAX' = \angle QAQ'$ as they are directly opposite angles. Hence, $\angle YAY' = \angle QAQ'$ and by respective addition of these angles to $\angle YAQ'$, it follows that $\angle Q'AY' = \angle QAY$. However, since $\angle AY'B = \angle AYB$ on chord *AB*, it follows that triangles *QAY* and *Q'AY'* are similar. Hence, the corresponding angles at *Q* and *Q'* are equal, and therefore lie on the same circular arc on fixed chord *AB*.

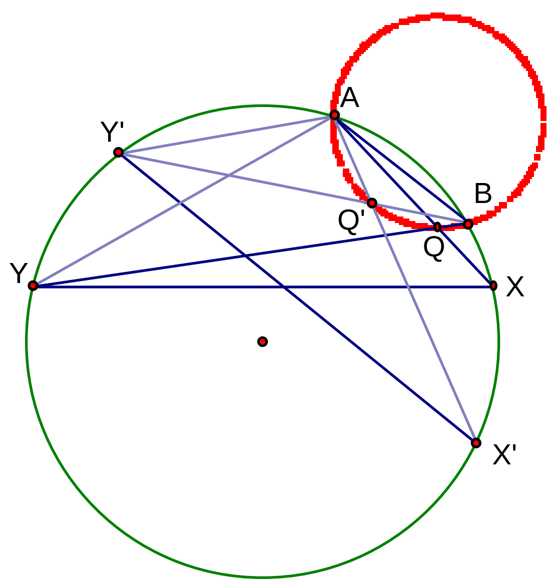

Figure 2

It is now left to the reader to verify that triangles *QAY* and *Q'AY'* are still similar in the position shown in Figure 2. Thus, the corresponding exterior angles at *Q* and *Q'* are equal, therefore also lying on a circular arc on fixed chord *AB*.

Since chords *XY* and *X'Y'* are equal, $\angle XAY = \angle X'AY'$ in both Figure 1 and 2, triangle *QAY* in Figure 1 is similar to triangle *QAY* in Figure 2 ($\angle AYB$ also remains constant on fixed *AB* in both figures). Thus, $\angle AQB$ in Figure 1 is supplementary to $\angle AQB$ in Figure 2, showing that the two circular arcs from the figures lie on the same circle.

Students' Solutions

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Instead of just presenting the problem, its generalization and solution directly to my students, the original problem was given as homework to the graduate students in mathematics education in my geometry class of Fall 2007.

Following up on my hint that they should try and prove $\angle XQY$ constant, Valerie Mckay showed using persistent angle chasing that $\angle XQY$ was indeed constant since it depended only on the fixed angles subtended at the centre of the circle by the chords of fixed lengths *XY* and *AB*. Another student, Jean Linner, then pointed out that essentially Valerie's solution was equivalent to the following theorem I'd not seen before (at least not stated as a theorem). She found it in a geometry book, and it directly explains why the result is true: "Given two (unequal) chords *AB* and *XY* with $AB < XY$, and *XA* and *YB* extended meet in *Q*, then $\angle XQY$ is half the difference of the intercepted arcs *XY* and *AB*, or equivalently, $\angle XQY = \frac{x-a}{2}$ "(see Figure 3). Proving this useful theorem is left as an exercise to the reader.

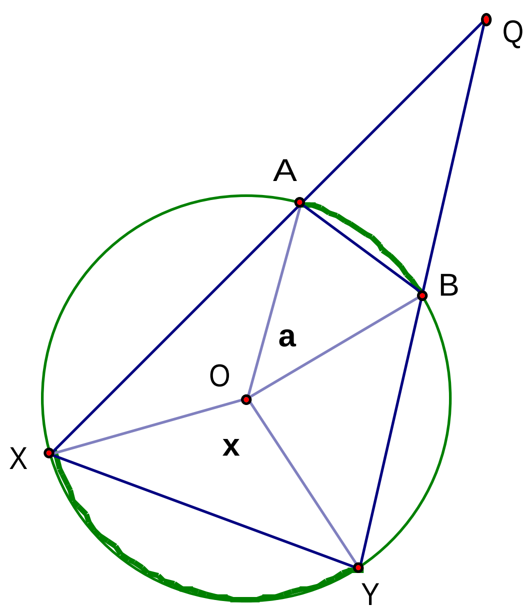

Figure 3

Notes

1. It might also be a good challenge for students to try and solve this problem using coordinate geometry.
2. A Dynamic Geometry (*Sketchpad 4*) sketch in zipped format (Winzip) of the

geometry results discussed here can be downloaded directly from: http://mysite.mweb.co.za/residents/profmd/circlelocus.zip

(If not in possession of a copy of *Sketchpad 4*, these sketches can be viewed with a free demo version of *Sketchpad 4* that can be downloaded from: http://www.keypress.com/x17670.xml)

3. An interactive *JavaSketchpad* sketch which can be manipulated without the demo is also available directly at: http://math.kennesaw.edu/~mdevilli/circlelocus.html

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