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Some Hexagon Area Ratios:

Problem Solving by Related Example

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> *Have you seen it before? Or have you seen the same problem in a slightly different form? Do you know a related problem? Do you know a theorem that could be useful? Look at the unknown! And try to think of a familiar problem having the same or a similar unknown. Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method?*
>
> George Polya (1945)

As starters, I would like to strongly encourage the reader to try to prove each of the three conjectures in this article before looking at my solutions as some readers may very well find even shorter, more elegant proofs than those give here. However, if the reader is not successful in finding a proof, s/he is likely to benefit far more from reading further than someone who hasn't tried at all.

Conjecture

One obvious way of generalizing the concept of a parallelogram to hexagons is to that of a *parallelo-hexagon* (a hexagon with opposite sides equal and parallel). Recently I was playing around with *Sketchpad* when I discovered the following interesting property of a parallelo-hexagon: if *G*, *H*, *I,* and *J* are the respective midpoints of the sides *AB*, *BC*, *DE* and *EF* of a parallelo-hexagon *ABCDEF*, then area *ABCDEF* = 2 area *GHIJ* (Figure 1).

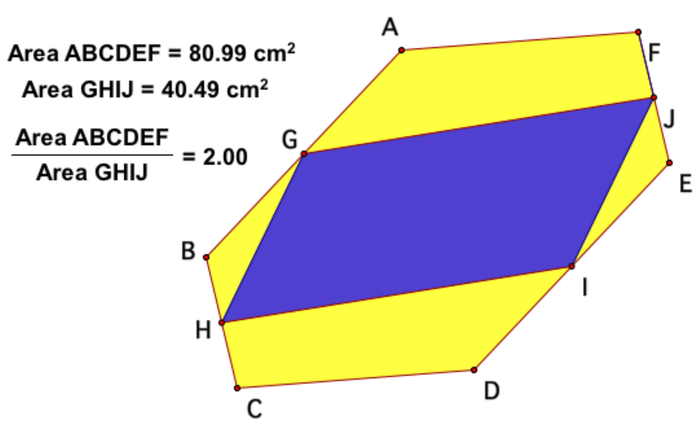

Figure 1

Dragging the parallelo-hexagon into different shapes, including into concave and crossed cases, quickly convinced me of the generality of the result. But why was it true? Could I derive an explanatory proof for it?

Related problem

The problem immediately reminded me of a similar related problem, namely, Varignon's Theorem, which states that the midpoints of the sides of any quadrilateral (including crossed ones) form a parallelogram half the area of the original quadrilateral. In De Villiers (2003) two different proofs are given (see Figure 2).

The one proof uses the easily-demonstrated fact that area ΔAEH = ¼ area of ΔABD, and then writing the area of the other three 'outer' triangles similarly, one simply subtracts the area of these triangles from that of the whole quadrilateral *ABCD*, and rearrange to get the desired result. The other proof is more visual and involves translating the parallelogram *EFGH* by vector *EH* as shown, then demonstrating that the sum of the areas of the triangles outside *EFGH* equals the area of the translated parallelogram *HGG′H′*. For example, $\Delta HDH'$ is congruent to ΔHAE, $\Delta GDG'$ is congruent to ΔGCF, $\Delta H'DG$ is congruent to triangle ΔEBF, from which the result then follows.

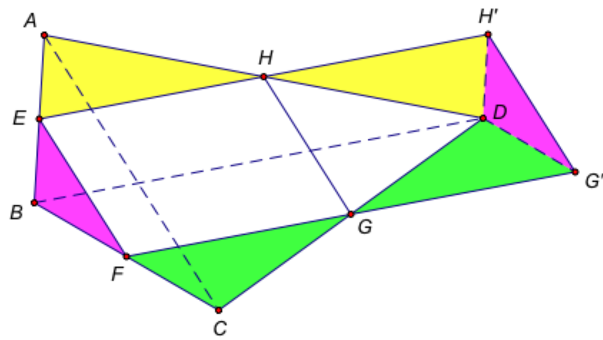

Figure 2

Considering Figure 1, like in the first proof of the Varignon area result, we can easily see that area ΔBHG = ¼ area ΔBCA, but then we get stuck writing the area of *CDIH* in terms of area *CDEB*. So let us try the other approach.

Proof

The parallelo-hexagon *ABCDEF* obviously has half-turn symmetry since it has opposite sides equal and parallel. From symmetry of the constructed midpoints, it follows that *GHIJ* also has half-turn symmetry, and *GHIJ* is therefore a parallelogram.

Here, for the sake of simplicity, we will consider only proofs of the convex and concave cases. Translate parallelogram *GHIJ* by vector *GH* as shown in Figure 3 for a convex and a concave case. We now have by the side-angle-side congruency condition, $\Delta HCH'$ congruent to ΔHBG and $\Delta IDI'$ congruent to ΔIEJ. Since both corresponding sides and angles are respectively equal, it follows that *CH′I′D* is also congruent to *AGJF*. In the convex case, we now have the areas of $\Delta HCH'$, *HCDI*, *CH′I′D,* and $\Delta IDI'$ adding up to the area of *HH′I′I*, from which the required result follows. Similarly, in the concave case, the sum of the areas of *HCDI* and *CH′I′D* minus the sum of the areas of $\Delta HCH'$ and $\Delta IDI'$ gives the desired result.

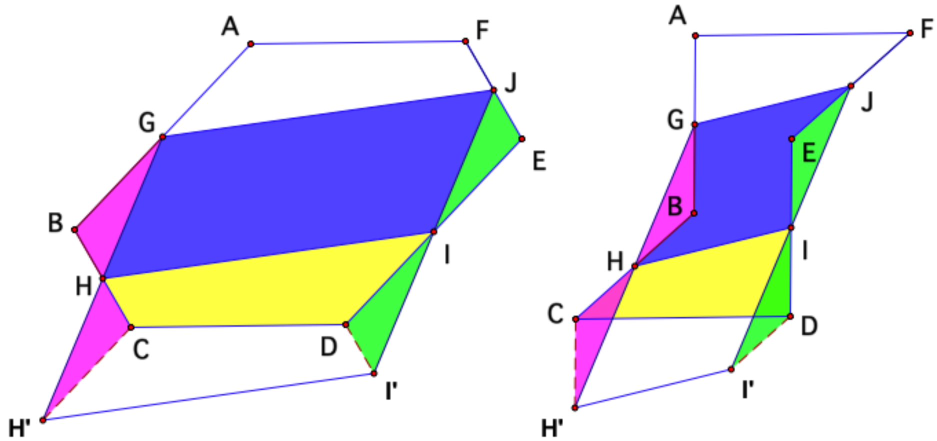

Figure 3

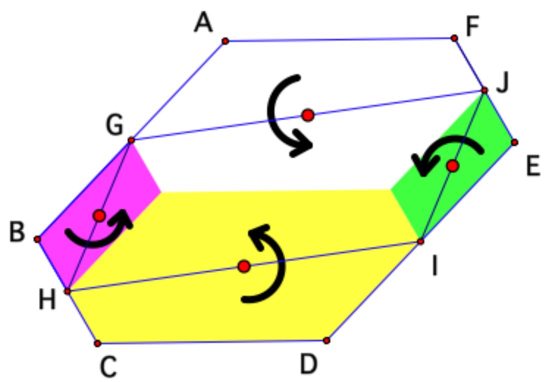

Figure 4

Looking back

Further reflection on the result shows that we can perhaps more easily demonstrate the result by giving each of the two triangles and the two quadrilaterals outside *GHIJ* a half-turn around the respective midpoints of the sides of *GHIJ* as shown for the convex parallelo-hexagon in Figure 4 to show that they cover *GHIJ*.

Another conjecture

As pointed out by Polya and others, answers to mathematical problems are seldom the end, but frequently prompt the beginning of new questions. So it's natural to ask: what happens to the area ratios if we connect the midpoints of all the sides of a parallelo-hexagon? Investigation with *Sketchpad* quickly shows that in this case the area of the formed hexagon *GHIJKL* is ¾ the area of *ABCDEF*.

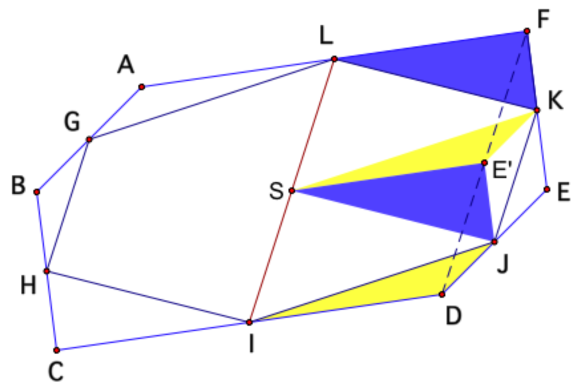

Figure 5

Proof

Consider the convex hexagon *ABCDEF* shown in Figure 5 (the concave case is left to the reader). In this case, we now want to prove that the sum of the areas of the triangles outside *GHIJKL* = 1/3 area *GHIJKL* to complete the proof.

Firstly note as before, from symmetry, that *GHIJKL* is also a parallelo-hexagon. Next note that *KJ* // *FD* and *KJ* = ½ *FD*, but *FD* // *LI* and *FD* = *LI* is given by construction. Hence, *KJ* // *LI* and *KJ* = ½ *LI*. This implies *KJ* is parallel and equal to both *LS* and *SI*, where *S* is the midpoint of *LI*. Now translate Δ*IDJ* by vector *JK* to map to Δ*SE´K*, and Δ*LFK* by vector *KJ* to map to Δ*SE´J* as shown. Since *E´JEK* is a parallelogram, a half-turn of Δ*JEK* around the midpoint of *KJ* would map it onto Δ*KE´J*.

We now have that the sum of the areas of triangles *IDJ*, *JEK* and *LFK* equal to the area of triangle *SKJ*. So the area of parallelogram *LSJK* equals twice the sum of the areas of triangles *IDJ*, *JEK* and *LFK*. But from symmetry, triangles *IDJ*, *JEK* and *LFK* are congruent to the corresponding triangles directly opposite them, and also outside *GHIJKL*, and hence the area of parallelogram *LSJK* equals the sum of the areas of all the triangles outside *GHIJKL*.

But parallelogram *LSJK* is equal in area to parallelogram *SIJK* (same base *JK* and between same parallels). However, parallelogram *SIJK* is equal in area to parallelogram *HIJS* (same base *IJ* and between same parallels), which in turn can be shown by continuing in the same way to be equal in area to parallelogram *GHSL*. In other words, *GHIJKL* is subdivided into parallelograms *LSJK*, *HIJS* and *GHSK*, all of equal area. This shows that the area of parallelogram *LSJK* = 1/3 area of *GHIJKL*, and completes the proof for the convex case.

Still another conjecture

But what happens to the area ratios if we connect the midpoints of the alternate sides of a parallelo-hexagon? Investigation with *Sketchpad* quickly shows that in this case the area of the formed triangle Δ*GHI* is 3/8 the area of *ABCDEF*.

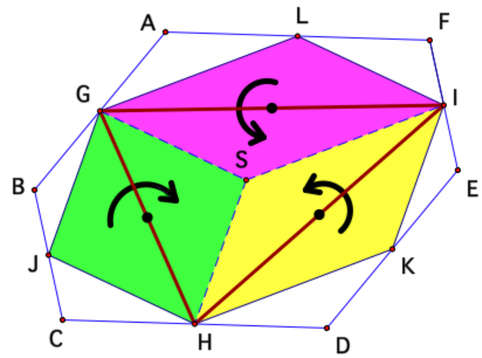

Figure 6

Proof

Perhaps not surprisingly, it is easy to see visually as shown in Figure 6 for a convex parallelo-hexagon, why the area of *GHI* is half that of the parallelo-hexagon *GJHKIL*. But from the preceding result we saw that *GJHKIL* has area ¾ that of *ABCDEF*, and thus proves the result.

Explore More

It seems natural to expect the above results to generalize to similar constant area ratios for inscribed polygons of a parallelo-octagon, but a quick inspection by dynamic geometry software will convince the reader that this not the case.

Concluding comments

Hopefully this article has shown that problem solving does not occur in a vacuum, but one is frequently required to draw on one's toolbox of accumulated past experiences of solving problems. It is often useful to recall proof techniques of related problems previously done as they may hold the key to opening the door to a proof of a result one is currently exploring. For example, in the first conjecture, we saw how translating the parallelogram similarly to the case for Varignon's theorem, immediately brought success. Further reflection led to realizing that the same result could be obtained differently by using half-turns on the outer polygons, and continuing in a similar vein, proofs were constructed for the other two conjectures.

Note that we didn't use Varignon's theorem itself, but that it was its proof that was useful in tackling these new conjectures. It is precisely for this reason that Yehuda Rav (1999) has eloquently, and perhaps provocatively, argued it is proofs rather than theorems that are the bearers of mathematical knowledge:

> *Theorems are in a sense just tags, labels for proofs, summaries of information, headlines of news, editorial devices. The whole arsenal of mathematical methodologies, concepts, strategies and techniques for solving problems, the establishment of interconnections between theories, the systematization of results — the entire mathematical know-how is embedded in proofs… Think of proofs as a*

> *network of roads in a public transportation system, and regard statements of theorems as bus stops; the site of the stops is just a matter of convenience.*

Consequently, Rav argues that proofs that provide useful methods, powerful tools and generalizable concepts for solving problems should be the primary focus of mathematical study, not so much theorems as such. In discussing Rav's paper, Hanna & Barbeau (2008) give two insightful case studies of mathematical examples of proof at the secondary-school level that lend themselves to the introduction of useful mathematical methods, tools, strategies and concepts.

It is hoped that this paper has similarly and modestly contributed a third case, which even though the results may not fall into the main stream curriculum at school, is easily accessible for talented high school students at the mathematics competition level. More generally, the strategy of transforming a figure, by 'cutting up and pasting in' parts of the figure by using transformations to show areas equivalent, is a useful strategy that features in many places, including finding a formula for the area of a triangle, parallelogram or trapezium, and even in many proofs of the theorem of Pythagoras.

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