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A generalization of Apollonius' theorem

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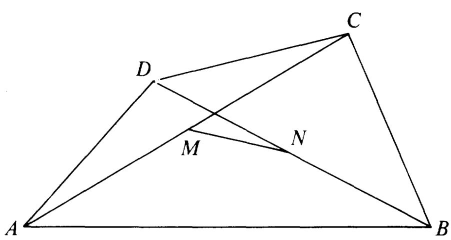
It follows at once from Pythagoras' theorem about a right-angled triangle that the sum of the squares of the lengths of the diagonals of a rectangle is equal to the sum of the squares of the lengths of the four sides. Apollonius showed that the assertion holds for a parallelogram and, more recently, Amir-Moez and Hamilton [1] gave a generalization to quadrilaterals by introducing a correction term which depends on the distance between the mid-points of the diagonals. In fact, they prove that if $ABCD$ is a quadrilateral and M, N are the mid-points of the diagonals, then

$$(AB)^2 + (BC)^2 + (CD)^2 + (DA)^2 = (AC)^2 + (BD)^2 + 4(MN)^2.$$



We now carry the generalization a stage further. Let E denote n-dimensional Euclidean space with inner (or scalar) product (,) and usual norm $\| \ \|$; so that if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, then

$$(\boldsymbol{x}, \boldsymbol{y}) = x_1 y_1 + \ldots + x_n y_n$$

and

$$\|\boldsymbol{x}\| = \mathscr{S}(\boldsymbol{x}, \boldsymbol{x}) = \sqrt{(x_1^2 + \ldots + x_n^2)}.$$

If A, B are points in E and $\boldsymbol{a}, \boldsymbol{b}$ are the corresponding vectors relative to some origin, then the distance AB from A to B is defined to be $\|\boldsymbol{b} - \boldsymbol{a}\|$, so that

$$(AB)^2 = \|\boldsymbol{b} - \boldsymbol{a}\|^2 = (\boldsymbol{b} - \boldsymbol{a}, \boldsymbol{b} - \boldsymbol{a}) = (\boldsymbol{b}, \boldsymbol{b}) + (\boldsymbol{a}, \boldsymbol{a}) - 2(\boldsymbol{b}, \boldsymbol{a})$$

by the bilinearity of the inner product.

Suppose now that $A_1, \ldots, A_{2k}$ are points of E and that $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2k}$ are the corresponding vectors. Let B, C denote respectively the orthocentres of the k-gons $A_2A_4 \ldots A_{2k}, A_1A_3 \ldots A_{2k-1}$, i.e. B, C are the points with vectors

$$\frac{1}{k}(\boldsymbol{a}_2 + \boldsymbol{a}_4 + \ldots + \boldsymbol{a}_{2k}),\ \frac{1}{k}(\boldsymbol{a}_1 + \boldsymbol{a}_3 + \ldots + \boldsymbol{a}_{2k-1}).$$

We denote by S the sum

$$\begin{aligned}
&(A_1A_2)^2 + (A_2A_3)^2 + \ldots + (A_{2k-1}A_{2k})^2 + (A_{2k}A_1)^2\\
&- (A_1A_3)^2 - (A_2A_4)^2 - \ldots - (A_{2k-1}A_1)^2 - (A_{2k}A_2)^2\\
&+ (A_1A_4)^2 + (A_2A_5)^2 + \ldots + (A_{2k-1}A_2)^2 + (A_{2k}A_3)^2\\
&- \ldots\\
&+ (-1)^k\{-A_1A_k)^2 + (A_2A_{k+1})^2 + \ldots + (A_{2k-1}A_{k-2})^2 + (A_{2k}A_{k-1})^2\}\\
&+ (-1)^{k+1}\{(A_1A_{k+1})^2 + (A_2A_{k+2})^2 + \ldots + (A_kA_{2k})^2\}.
\end{aligned}$$

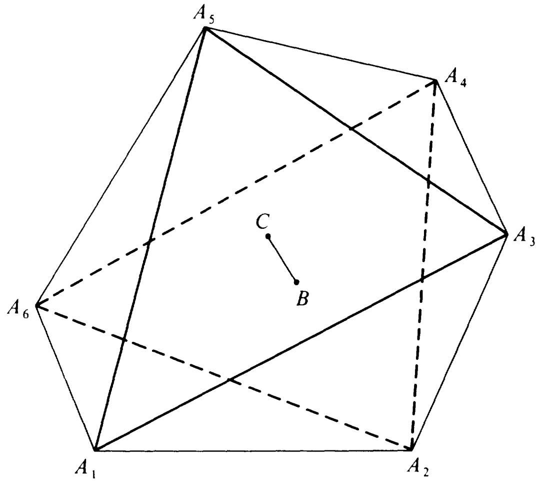
(Notice that the last row contains only k summands.)

THEOREM. $S = k^2(BC)^2$.

PROOF. Interpreting $\boldsymbol{a}_p$ as $\boldsymbol{a}_{p-2k}$ when $p > 2k$, we have

$$\begin{aligned}
S &= \sum_{j=1}^{k-1}\{(-1)^{j+1}\sum_{i=1}^{2k}(\boldsymbol{a}_{i+j} - \boldsymbol{a}_i, \boldsymbol{a}_{i+j} - \boldsymbol{a}_i)\}\\
&\qquad + (-1)^{k+1}\sum_{i=1}^{k}(\boldsymbol{a}_{i+k} - \boldsymbol{a}_i, \boldsymbol{a}_{i+k} - \boldsymbol{a}_i)\\
&= \sum_{i=1}^{2k}(\boldsymbol{a}_i, \boldsymbol{a}_i) - 2\sum_{j=1}^{k-1}\{(-1)^{j+1}\sum_{i=1}^{2k}(\boldsymbol{a}_{i+j}, \boldsymbol{a}_i)\} - 2(-1)^{k+1}\sum_{i=1}^{k}(\mathbf{a}_{i+k}, \mathbf{a}_i)\\
&= \left(\sum_{i=1}^{2k}(-1)^{i+1}\boldsymbol{a}_i, \sum_{i=1}^{2k}(-1)^{i+1}\boldsymbol{a}_i\right) = \left\|\sum_{i=1}^{2k}(-1)^{i+1}\boldsymbol{a}_i\right\|^2\\
&= k^2\left\|\frac{\boldsymbol{a}_1 + \boldsymbol{a}_3 + \ldots + \boldsymbol{a}_{2k-1}}{k} - \frac{\boldsymbol{a}_2 + \boldsymbol{a}_4 + \ldots + \boldsymbol{a}_{2k}}{k}\right\|^2\\
&= k^2(BC)^2.
\end{aligned}$$

SPECIAL CASES. The case $k = 2$ is that given by Amir-Moez and Hamilton. For $k = 3$, we have



$$\begin{aligned}&(A_1A_2)^2+(A_2A_3)^2+(A_3A_4)^2+(A_4A_5)^2+(A_5A_6)^2+(A_6A_1)^2\\&-(A_1A_3)^2-(A_2A_4)^2-(A_3A_5)^2-(A_4A_6)^2-(A_5A_1)^2-(A_6A_2)^2\\&+(A_1A_4)^2+(A_2A_5)^2+(A_3A_6)^2\\&=9(BC)^2.\end{aligned}$$

Remarks on the Proof of the Theorem

1. The Proof holds for any inner product space.
2. The points $A_1, \ldots, A_{2k}$ can be taken quite arbitrarily; in particular, they need not be distinct or coplanar, and, if they are coplanar, then the polygon $A_1 \ldots A_{2k}$ need not be convex.

Apollonius' theorem says that $S = 0$ for parallelograms, but it is easy to find hexagons with

(a) opposite sides equal or
(b) alternate sides equal or
(c) opposite sides parallel

and $S \neq 0$ in each case.

We conclude by describing the construction of a general class of $2k$-gons for which $S = 0$. Take k points in E with vectors $\boldsymbol{a}_2, \boldsymbol{a}_4, \ldots, \boldsymbol{a}_{2k}$, say, let $\lambda_1, \ldots, \lambda_k$ be real numbers with $\lambda_1 + \ldots + \lambda_k = 1$ and put

$$\begin{aligned}\boldsymbol{a}_1 &= \lambda_1\boldsymbol{a}_2 + \lambda_2\boldsymbol{a}_4 + \ldots + \lambda_k\boldsymbol{a}_{2k}\\ \boldsymbol{a}_3 &= \lambda_1\boldsymbol{a}_4 + \lambda_2\boldsymbol{a}_6 + \ldots + \lambda_k\boldsymbol{a}_2\\ &\vdots\\ \boldsymbol{a}_{2k-1} &= \lambda_1\boldsymbol{a}_{2k} + \lambda_2\boldsymbol{a}_2 + \ldots + \lambda_k\boldsymbol{a}_{2k-2}.\end{aligned}$$

Adding gives

$$\boldsymbol{a}_1 + \boldsymbol{a}_3 + \ldots \boldsymbol{a}_{2k-1} = \boldsymbol{a}_2 + \boldsymbol{a}_4 + \ldots + \boldsymbol{a}_{2k}.$$

Thus, the orthocentres B, C coincide and it follows from the Theorem that $S = 0$.

Reference

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Cyclic polygons and related questions

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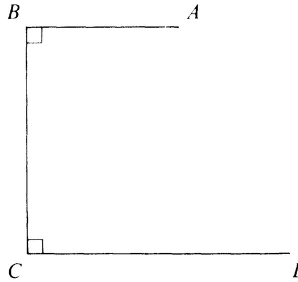
This is the story of a problem that began in an innocent way and in the course of time spread out in unexpected directions involving some interesting side-issues. It all began with the following problem which I constructed for the Mathematical Challenge competition run by the Scottish Mathematical Council.

Problem 1



Given three lines of lengths p, q, r, where $p < q < r$, arrange them to form the sides AB, BC, CD of a quadrilateral as shown, (with right angles at B and C), so that the quadrilateral $ABCD$ has maximum area. [The solution is