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H. B. GRIFFITHS

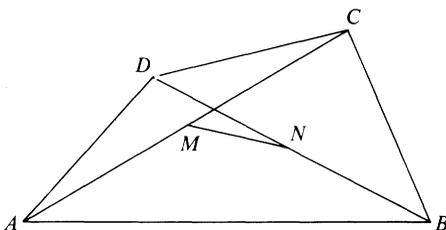
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## A generalization of Apollonius' theorem

A. J. DOUGLAS

It follows at once from Pythagoras' theorem about a right-angled triangle that the sum of the squares of the lengths of the diagonals of a rectangle is equal to the sum of the squares of the lengths of the four sides. Apollonius showed that the assertion holds for a parallelogram and, more recently, Amir-Moez and Hamilton [1] gave a generalization to quadrilaterals by introducing a correction term which depends on the distance between the mid-points of the diagonals. In fact, they prove that if  $ABCD$  is a quadrilateral and  $M, N$  are the mid-points of the diagonals, then

$$(AB)^2 + (BC)^2 + (CD)^2 + (DA)^2 = (AC)^2 + (BD)^2 + 4(MN)^2.$$



We now carry the generalization a stage further. Let  $E$  denote  $n$ -dimensional Euclidean space with inner (or scalar) product  $(\cdot, \cdot)$  and usual norm  $\|\cdot\|$ ; so that if  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$ , then

$$(x, y) = x_1 y_1 + \dots + x_n y_n$$

and

$$\|x\| = \mathcal{E}(x, x) = \sqrt{(x_1^2 + \dots + x_n^2)}.$$

If  $A, B$  are points in  $E$  and  $a, b$  are the corresponding vectors relative to some origin, then the distance  $AB$  from  $A$  to  $B$  is defined to be  $\|b - a\|$ , so that

$$(AB)^2 = \|b - a\|^2 = (b - a, b - a) = (b, b) + (a, a) - 2(b, a)$$

by the bilinearity of the inner product.

Suppose now that  $A_1, \dots, A_{2k}$  are points of  $E$  and that  $\mathbf{a}_1, \dots, \mathbf{a}_{2k}$  are the corresponding vectors. Let  $B, C$  denote respectively the orthocentres of the  $k$ -gons  $A_2 A_4 \dots A_{2k}, A_1 A_3 \dots A_{2k-1}$ , i.e.  $B, C$  are the points with vectors

$$\frac{1}{k}(\mathbf{a}_2 + \mathbf{a}_4 + \dots + \mathbf{a}_{2k}), \quad \frac{1}{k}(\mathbf{a}_1 + \mathbf{a}_3 + \dots + \mathbf{a}_{2k-1}).$$

We denote by  $S$  the sum

$$\begin{aligned} & (A_1 A_2)^2 + (A_2 A_3)^2 + \dots + (A_{2k-1} A_{2k})^2 + (A_{2k} A_1)^2 \\ & - (A_1 A_3)^2 - (A_2 A_4)^2 - \dots - (A_{2k-1} A_1)^2 - (A_{2k} A_2)^2 \\ & + (A_1 A_4)^2 + (A_2 A_5)^2 + \dots + (A_{2k-1} A_2)^2 + (A_{2k} A_3)^2 \\ & - \dots \\ & + (-1)^k \{ -A_1 A_k \}^2 + (A_2 A_{k+1})^2 + \dots + (A_{2k-1} A_{k-2})^2 + (A_{2k} A_{k-1})^2 \} \\ & + (-1)^{k+1} \{ (A_1 A_{k+1})^2 + (A_2 A_{k+2})^2 + \dots + (A_k A_{2k})^2 \}. \end{aligned}$$

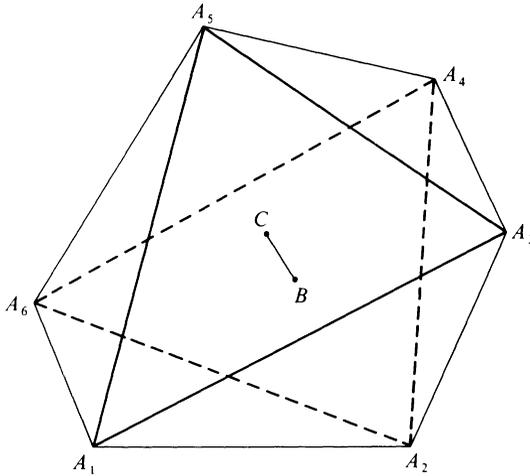
(Notice that the last row contains only  $k$  summands.)

**THEOREM.**  $S = k^2(BC)^2$ .

**PROOF.** Interpreting  $\mathbf{a}_p$  as  $\mathbf{a}_{p-2k}$  when  $p > 2k$ , we have

$$\begin{aligned} S &= \sum_{j=1}^{k-1} \{ (-1)^{j+1} \sum_{i=1}^{2k} (\mathbf{a}_{i+j} - \mathbf{a}_i, \mathbf{a}_{i+j} - \mathbf{a}_i) \} \\ & \quad + (-1)^{k+1} \sum_{i=1}^k (\mathbf{a}_{i+k} - \mathbf{a}_i, \mathbf{a}_{i+k} - \mathbf{a}_i) \\ &= \sum_{i=1}^{2k} (\mathbf{a}_i, \mathbf{a}_i) - 2 \sum_{j=1}^{k-1} \{ (-1)^{j+1} \sum_{i=1}^{2k} (\mathbf{a}_{i+j}, \mathbf{a}_i) \} - 2(-1)^{k+1} \sum_{i=1}^k (\mathbf{a}_{i+k}, \mathbf{a}_i) \\ &= \left( \sum_{i=1}^{2k} (-1)^{i+1} \mathbf{a}_i, \sum_{i=1}^{2k} (-1)^{i+1} \mathbf{a}_i \right) = \left\| \sum_{i=1}^{2k} (-1)^{i+1} \mathbf{a}_i \right\|^2 \\ &= k^2 \left\| \frac{\mathbf{a}_1 + \mathbf{a}_3 + \dots + \mathbf{a}_{2k-1}}{k} - \frac{\mathbf{a}_2 + \mathbf{a}_4 + \dots + \mathbf{a}_{2k}}{k} \right\|^2 \\ &= k^2(BC)^2. \end{aligned}$$

**SPECIAL CASES.** The case  $k = 2$  is that given by Amir-Moez and Hamilton. For  $k = 3$ , we have



$$\begin{aligned}
 & (A_1 A_2)^2 + (A_2 A_3)^2 + (A_3 A_4)^2 + (A_4 A_5)^2 + (A_5 A_6)^2 + (A_6 A_1)^2 \\
 & - (A_1 A_3)^2 - (A_2 A_4)^2 - (A_3 A_5)^2 - (A_4 A_6)^2 - (A_5 A_1)^2 - (A_6 A_2)^2 \\
 & + (A_1 A_4)^2 + (A_2 A_5)^2 + (A_3 A_6)^2 \\
 & = 9(BC)^2.
 \end{aligned}$$

*Remarks on the Proof of the Theorem*

1. The Proof holds for any inner product space.
2. The points  $A_1, \dots, A_{2k}$  can be taken quite arbitrarily; in particular, they need not be distinct or coplanar, and, if they are coplanar, then the polygon  $A_1 \dots A_{2k}$  need not be convex.

Apollonius' theorem says that  $S = 0$  for parallelograms, but it is easy to find hexagons with

- (a) opposite sides equal or
- (b) alternate sides equal or
- (c) opposite sides parallel

and  $S \neq 0$  in each case.

We conclude by describing the construction of a general class of  $2k$ -gons for which  $S = 0$ . Take  $k$  points in  $E$  with vectors  $\mathbf{a}_2, \mathbf{a}_4, \dots, \mathbf{a}_{2k}$ , say, let  $\lambda_1, \dots, \lambda_k$  be real numbers with  $\lambda_1 + \dots + \lambda_k = 1$  and put

$$\begin{aligned}
 \mathbf{a}_1 &= \lambda_1 \mathbf{a}_2 + \lambda_2 \mathbf{a}_4 + \dots + \lambda_k \mathbf{a}_{2k} \\
 \mathbf{a}_3 &= \lambda_1 \mathbf{a}_4 + \lambda_2 \mathbf{a}_6 + \dots + \lambda_k \mathbf{a}_2 \\
 &\vdots \\
 \mathbf{a}_{2k-1} &= \lambda_1 \mathbf{a}_{2k} + \lambda_2 \mathbf{a}_2 + \dots + \lambda_k \mathbf{a}_{2k-2}.
 \end{aligned}$$

Adding gives

$$\mathbf{a}_1 + \mathbf{a}_3 + \dots + \mathbf{a}_{2k-1} = \mathbf{a}_2 + \mathbf{a}_4 + \dots + \mathbf{a}_{2k}.$$

Thus, the orthocentres  $B$ ,  $C$  coincide and it follows from the Theorem that  $S = 0$ .

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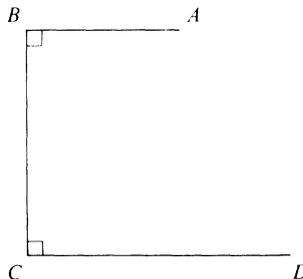
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## Cyclic polygons and related questions

D. S. MACNAB

This is the story of a problem that began in an innocent way and in the course of time spread out in unexpected directions involving some interesting side-issues. It all began with the following problem which I constructed for the Mathematical Challenge competition run by the Scottish Mathematical Council.

### Problem 1



Given three lines of lengths  $p$ ,  $q$ ,  $r$ , where  $p < q < r$ , arrange them to form the sides  $AB$ ,  $BC$ ,  $CD$  of a quadrilateral as shown, (with right angles at  $B$  and  $C$ ), so that the quadrilateral  $ABCD$  has maximum area. [The solution is