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Equality is not always 'best'!

Michael de Villiers

University of KwaZulu-Natal

profmd@mweb.co.za

Mathematical modeling as an important mathematical skill or process is emphasized in the 2011 Curriculum and Assessment Policy Statement (CAPS) for Mathematics in South Africa as follows:

> "*Mathematical modeling is an important focal point of the curriculum. Real life problems should be incorporated into all sections whenever appropriate. Examples used should be realistic and not contrived. Contextual problems should include issues relating to health, social, economic, cultural, scientific, political and environmental issues whenever possible.*" (Department of Basic Education, 2011, p. 11)

THE MODELING PROCESS

As discussed in De Villiers (2007), the process of mathematical modeling essentially consists of three steps or stages as illustrated in Figure 1, namely (i) construction of the mathematical model, (ii) solution of the model, and (iii) interpretation and evaluation of the solution. Traditionally, much of schooling consisted of teaching learners how to solve given mathematical models such as linear, quadratic, trigonometric, exponential, and hyperbolic functions, hardly involving them at all in the *construction* and *development* of these models from real world contexts, and much less on *evaluating* the solutions and how well they fit 'reality'. In short, it consisted of teaching mathematics largely disconnected from the real world.

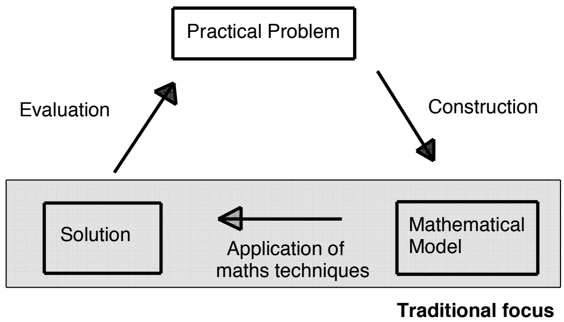

FIGURE 1: Three stages of mathematical modeling.

In traditional geometry teaching at the GET (Grades 7-9) and FET (Grades 10-12) Mathematics levels in South Africa, this disconnection has been even worse, with teaching and textbooks ignoring applications almost completely, and to a large extent only developing the geometry content with the idea to engage learners with 'rigorous' proof in a deductive system. So the intent of the new CAPS curriculum to promote mathematical modeling is welcome, and at last bringing South Africa more in line with trends in other countries. However, despite the laudable statements about mathematical modeling in the new Mathematics CAPS quoted at the beginning of this article, one wonders to what extent a genuine effort will be made to connect geometry to the real world, especially since the CAPS documents do not provide any examples of possible real world contexts.

In all the university courses I teach to prospective mathematics teachers, I try to bring in mathematical modeling as much as possible and constantly expect students not only to develop their own mathematical models, but also critically to evaluate the sense of the solutions provided by their models.

EVALUATION OF A SOLUTION

In Makae et al. (2001, pp. 171-180) and De Villiers (2003, pp. 27-33) the concept of 'perpendicular bisector' is introduced as 'a locus of equidistant points from the endpoints of a line-segment' in order to find suitable equidistant points from four and three rural villages[1] respectively to build a water reservoir. Associated with solving these two problems is the simultaneous introduction of the concepts 'circumcentre' (as a point, if it exists, equidistant from $n > 2$ points) and 'cyclic polygons' (as polygons whose perpendicular bisectors of the sides are concurrent at the circumcentre; and therefore their vertices lie on a circle). Instead of using congruency as in the traditional proof, the concept of equidistance is then also used to logically *explain* (prove) *why* the perpendicular bisectors are concurrent for a triangle.

During 2011 I was in class with my prospective mathematics teacher students reflecting on the investigation for four and three villages that the students had completed in the previous session in the computer lab. I had open on my computer, and projected on screen, a triangle (ABC) with circumcentre (O) constructed with *Sketchpad* as the required equidistant point for the reservoir. To revise a question on the worksheet I dragged the triangle into a right-angled triangle and then continued dragging until it became obtuse, so that the circumcentre (O) moved far outside the triangle formed by the 3 villages (A, B and C) as shown in Figure 2.

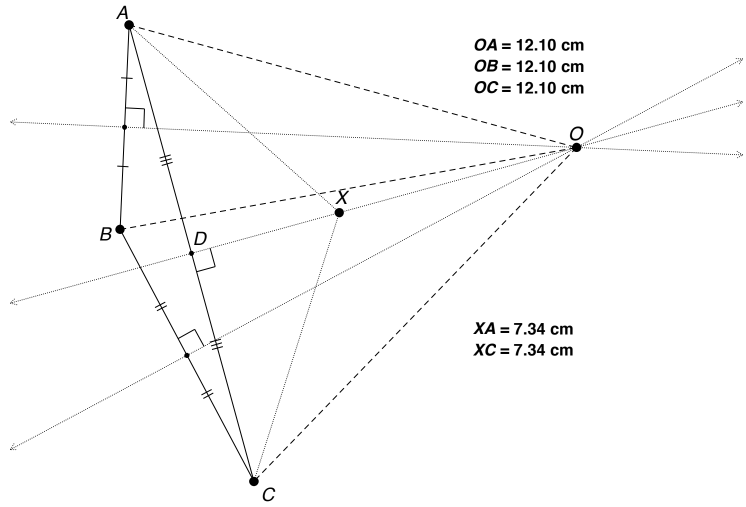

FIGURE 2: Circumcentre O outside triangle ABC.

I then repeated the question from the worksheet: "*What did you notice about the position of the circumcentre … the desired position for the reservoir … when the triangle is acute, right and obtuse?*" to which students quickly responded that it would for each case respectively lie inside the triangle, at the midpoint of the hypotenuse, and outside the triangle. Then I remarked, gesturing to the projected diagram for the obtuse triangle, something to the effect of: "*Isn't it surprising that the equidistant point for building the reservoir in the obtuse case lies completely outside the triangle?*" At that point, one of my students, Xolile, shook his head and responded: "*Eish! But that would not be a good position to build the water reservoir …*" And immediately I responded by shrugging and saying: "*But that's the circumcentre … the unique point exactly the same distance from all three villages, don't you agree? … as was the requirement of the problem right at the beginning. That is fair, isn't it: that the villagers from each village walk equal distances?*" However, Xolile continued to shake his head and said: "*Eish, but then they have to walk too far …*" "*Yes, Xolile*", I agreed. "*They obviously now have to walk further than when the triangle is acute. But they are all still equal distances from the reservoir; that's only fair, isn't it?*" However, not wanting to just dismiss him, I paused for a moment and asked: "*Well, where do you feel the reservoir should instead be built when the triangle is obtuse?*" To this, Xolile responded: "*I think the reservoir should not be built outside the triangle because then they have to walk too far …*"

Only then did it hit me what he meant, and that he was perfectly right! It was visually obvious in the diagram! Something neither I nor any student from previous classes had raised before or critically reflected upon. If distance was the over-riding factor, the circumcentre was perhaps not the 'best' choice in the obtuse case. Even though the circumcentre was the 'fair' solution in being equidistant from all three villages, it seemed rather silly to have to walk so far for water!

[1] It was assumed at the outset that the villages were all of equal size, while students were expected to identify other assumptions such as that the villages were on the same plane, that the surface was flat, that there were no natural obstacles such as dense forest or deep dongas, and that the distances were sufficiently large in comparison to the sizes of the villages and the water reservoir that they could geometrically be represented by points.

In the subsequent discussion with the students, the following 'better real world' solution was arrived at. Consider any point X on the perpendicular bisector of AC as shown in Figure 2. Clearly, the closer X comes to the triangle, the shorter the equal distances (AX and CX) from villages A and C to X will become, and these distances will become a minimum when X reaches the midpoint D of AC, and would increase again if X went past D. Since the midpoint D results in the shortest possible equal distances between villages A and C, midpoint D would obviously be the preferred choice for the reservoir between these two villages[2]. Moreover, village B will be quite happy with that choice and not complain, because D is much closer for them than building the reservoir at O. Of course, the villagers from village B will be advantaged over the villagers from the other two in walking a shorter distance to the water[3]. But if villages A and C complain and put their foot down and insist on absolute equality and fairness, then they will have no choice but to have the reservoir built at O. But this will mean that everyone will have to walk much further than the choice at D! So **only if** they are willing to *cooperate* and ***compromise*** on the absolute insistence of equality in terms of distance will they benefit and be able to choose D - with the result that everyone would be walking a substantially shorter distance. So perhaps surprisingly in this case, equality is not the 'best' choice they could make, and agreeing instead to 'violate' equality would provide an improved 'real world' solution!

SOME FURTHER EXTENSIONS AND EXPLORATIONS

Cyclic quadrilateral

In the same way as above, we could argue that in the case of four villages of equal size forming a cyclic quadrilateral[4], the 'best' choice of the water reservoir when the circumcentre O lies outside the cyclic quadrilateral (as in Figure 3) would lie at the midpoint E of the longest side BC.

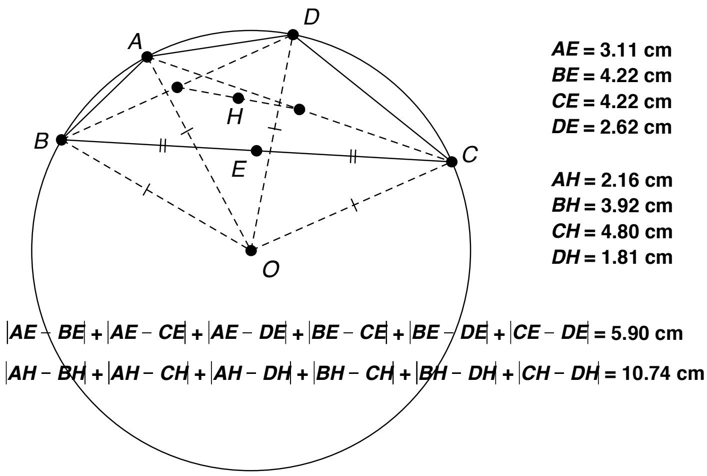

FIGURE 3: Circumcentre O outside cyclic quadrilateral formed by villages A, B C and D.

However, someone might persuasively argue that if one just considers obtuse ΔABD, then the best choice would be the midpoint of BD. Similarly, the best choice for obtuse ΔADC is at the midpoint of AC. Wouldn't a better choice for the reservoir perhaps be the midpoint H of these two midpoints rather than E? After all, if one looks at the displayed measurements, three of the villages (A, B and D) would have shorter walking distances than to E, and only village C would have a longer walking distance than before. Doesn't that seem more 'fair'? However, if we examine and compare the distances more closely, it is immediately noticeable that the walking distance from village C to H is more than twice the walking distances from villages A and D (and the walking distance of village B is also substantially more than those from villages A and D). In contrast, when we look at the walking distances from the four villages to E and compare them, we 'see' that they are much closer in value. In other words, there's a bigger

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[2] Formally, $DC < XC$ because XC is the hypotenuse of right ΔXDC (which itself follows from the theorem that the longest side in a triangle is opposite the largest angle), with the minimum of XC clearly occurring when X coincides with D.

[3] Though visually 'obvious' from the given diagram, $BD < AD$, since angle BAD < angle ABD.

[4] Unlike triangles, not all quadrilaterals are cyclic; so if the 4 villages do not form a cyclic quadrilateral the 'best' choice of position can be found by minimizing the sum of the differences (pairwise) between all the distances from the reservoir to the 4 villages (see De Villiers, 2003, p. 151). This can be done by measurement and dragging in *Sketchpad*, or algebraically, by applying the least squares method.

'difference' or 'discrepancy' between the comparative walking distances from the villages to H than between the comparative walking distances from the villages to E.

Mathematically, we can quantify the above by finding the sum of the absolute values[5] of the differences between all the distances from the villages to respectively points H and E to see which one gives the minimum value. As shown by the calculations in Figure 3, it's clear that this 'total discrepancy' between the walking distances is less for point E than H. Hence, E ought to be the preferred choice.

Building pipelines

The context of building a water reservoir for 3 or 4 (or more) villages provides a rich context for mathematical modeling and problem solving. In Makea et al. (2001, pp. 181-186) the original problem is extended to selecting representatives from the four villages onto a Water Board, providing an opportunity to discuss some elementary mathematical issues surrounding 'apportionment' in order to obtain 'fair representation' if the four villages are of different sizes (also compare Nielsen & De Villiers, 2012).

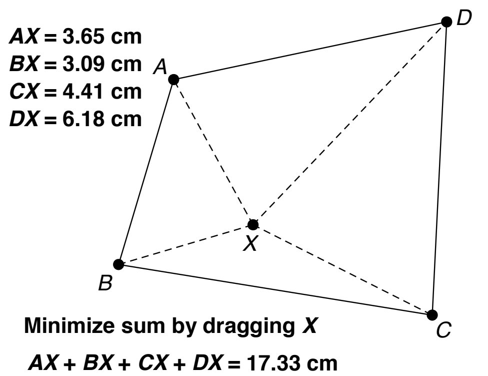

FIGURE 4: Minimizing the sum of the distances.

Suppose next a nature conservationist on the eventually elected Water Board suggests that perhaps it might be better for the environment to rather build pipelines from the water reservoir to the villages, as not only would that save the villagers the effort of walking, but also prevent the formation of footpaths that can lead to serious vegetation and soil erosion (compare Makea et al., 2001, pp. 187-189). From a cost perspective, the problem now translates into *minimizing the sum of the distances* from the water reservoir to the villages (see Figure 4).

From several experiences with learners at both school and university (undergraduate and postgraduate) I've noted that when confronted with this problem they initially tend to believe that the 'best' solution would still be to try and find an equidistant point from the four villages (thereby presupposing the existence of a circumcentre that would of course only exist if the four villages were concyclic). It thus usually comes to them as a great surprise and learning experience when, investigating dynamically, they find that the solution for a (convex) quadrilateral lies at the intersection of the diagonals[6] (which is then easily explained (proved) using the triangle inequality).

Instead of a water reservoir, Hähkiöniemi, Leppäaho and Francisco (2011, p. 31) used the context of building an amusement park together with an open-ended approach with Grade 9 learners as follows:

> "Four towns plan to build together a magnificent amusement park. Investigate using *GeoGebra* what would be the most optimal and fair location for the amusement park."

In such an 'open' approach learners are free to suggest and propose the kind of solution they consider the 'best'. Interestingly, some learners came up with the equi-distant (circumcentre) proposal, while others suggested mimimizing the sum of the distances to the amusement park.

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[5] Instead of taking absolute values, the differences could be squared (as in the 'least squares' method in statistics). The reason for making all the differences positive is to avoid having negative differences and positive differences canceling each other out.

[6] For a concave quadrilateral when the diagonals fall outside, the minimum sum of distances to the vertices is found at the vertex with the reflex angle (compare Makae et al., 2001, pp. 188-189).

Going 3D

Lastly, one could also explore the analogous case of finding a suitable equidistant point for say a space-station for 4 planets in space forming a tetrahedron. Such an activity would naturally lead to the discovery (and proof) that, just like the triangle, the tetrahedron has a circumcentre and an associated circumsphere (compare Camou et al., 2013, pp. 68-71). However, now a similar problem as with the triangle arises when the tetrahedron becomes obtuse and the circumcentre falls outside the tetrahedron (see Figure 5). Since area in 3D is the analogue of length in 2D, we might, analogously to the obtuse triangle, expect the 'best' solution for an obtuse tetrahedron to lie at the circumcentre of the largest triangle lying opposite the largest dihedral angle. But what happens if that triangle itself is also obtuse? Does the 'best' solution then lie at that triangle's longest side (edge)? These questions are deliberately left open to the reader to investigate further. A dynamic tool like *Cabri 3D* might be useful to conduct such an exploration.

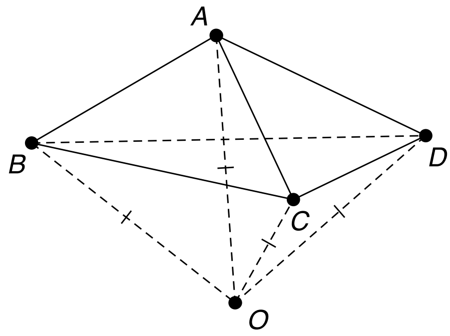

FIGURE 5: Four planets forming a tetrahedron.

CONCLUDING REMARKS

The episode with Xolile and the obtuse triangle and circumcentre rather nicely demonstrates that not only is mathematics never a precise model of the real world due to various simplifying assumptions that inevitably have to be made, but also that other practical considerations may indicate various different options that might be 'better' than the 'purely' mathematical solution. In contrast to 'pure' mathematics and its 'neat' solutions, 'applied' mathematics tends to be more 'untidy' with the need to interpret solutions given by mathematical models in a practical light. The episode also provides a good pedagogical example of how our students may sometimes teach us something new, provided we give them the opportunity to voice their opinions, and really listen to and try to understand what they have to say, especially if it doesn't agree with our own pre-conceived ideas.

REFERENCES

Camou, B., Olive, J, Colucci, M., & Garcia, G. (2013). *Essential 3D Geometry.* San Diego, CA: University Readers.

Department of Basic Education. (2011). *Curriculum and Assessment Policy Statement (CAPS): Mathematics.* Pretoria: DBE.

De Villiers, M. (2003). *Rethinking Proof with Sketchpad.* Emeryville: Key Curriculum Press.

De Villiers, M. (2007). Mathematical applications, modeling and technology. In M. Setati et al. (Eds.), *Proceedings of the 13th Annual National Congress of AMESA* (pp. 47-64). Wits University: AMESA. (Available from: http://frink.machighway.com/~dynamicm/mathapplications.pdf)

Hähkiöniemi, M., Leppäaho, H., & Francisco, J. (2011). Model for teacher-assisted technology enriched open problem solving. In T. Bergqvist (Ed). *Learning problem solving and learning through problem solving. Proceedings from the 13th ProMath conference* (pp. 30-43). Umeä, UMERC.

Makae, E., Nhlapo, M., De Villiers, M., Glover, H., Shembe, M., & Masimela, M. (2001). *Let's talk mathematics! Learner's book Grade 9.* Hout Bay, South Africa: Ace Publishers.