Published in Mathematics in School, Sept 2007, Vol. 36, No. 4, pp. 9-11.

An example of the discovery function of proof

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"... (for) the future mathematician ... the most important part of the work is to look back at the completed solution. Surveying the course of his work and the final shape of the solution, he may find an unending variety of things to observe ... He should solve problems, ... meditate upon their solution, and invent new problems." - George Polya (1945:205-206)

Traditionally, the purpose of proof in the mathematics classroom and textbooks has been presented virtually exclusively as that of verification; i.e. as a means of obtaining certainty and to eliminate doubt. However, proof has many other important functions within mathematics, which in certain situations are often of far greater importance than that of mere verification. Some of these other functions are (see De Villiers 2003):

- explanation (providing insight into why it is true)
- discovery (the discovery or invention of new results)
- intellectual challenge (the self-realization/fulfilment derived from constructing a proof)
- systematisation (the organisation of various results into a deductive system of axioms, concepts and theorems)

For students both at high school and university to develop a holistic perspective and understanding of the role of proof, they should also be made aware, in some way, of these other functions of proof through suitably designed activities. As indicated by studies such as Mudaly & De Villiers (2000) and Govender & De Villiers (2004), some of these functions can be meaningfully experienced and successfully communicated to students.

Several new textbooks around the world today proclaim to using an "*investigative approach*" by which they mean that they try to accurately reflect how mathematicians conduct their research and make new advances. Unfortunately, upon closer analysis the majority of these books let students discover mathematical results experimentally, and then proof is introduced only as a means of "*making sure*" these experimentally discovered results are generally true. In other words, only the "*verification*" function of proof is really developed or introduced.

However, to the working mathematician proof (or more generally, deductive reasoning) is not merely a means of verifying an already-discovered result, but often also a means of exploring, analysing, discovering and inventing new results. Quite often new discoveries can and are made purely by analysing a problem deductively and analytically rather than experimentally. Moreover, sometimes a proof provides valuable insight into why the result is true, immediately enabling one to generalise or vary the result in different ways. This process corresponds exactly to the last "looking back" or "reflective" stage of Polya's famous model of problem solving (Polya 1945).

The purpose of this article is to give an illustrative example of the latter discovery function of proof, which might be accessible to high school students as well as prospective and in-service teachers. The author has found the problem and subsequent discussion reasonably effective to develop some appreciation of the discovery function during a recent Olympiad problem solving workshop with 30 high school teachers in KwaZulu-Natal, South Africa.

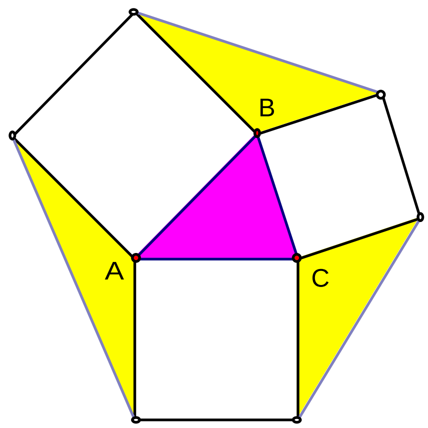

Figure 1: Prove that the areas of the four shaded triangles are the same.

A short while ago the author came across the interesting problem in Figure 1, which was posed by Faux (2004) to readers of the *Mathematics Teaching* journal, leaving it to the reader to infer from the figure that the three white quadrilaterals were squares. In the UK, this result is apparently called Cross's theorem, named after a 14 year old boy David Cross who "*discovered*" it. It is always nice when one's students make such "*new*" discoveries. For the student it provides a boost of confidence, and acts as a much stronger motivation for finding a proof than for the usual "*run of the mill*" theorems or riders that appear in class textbooks. Recently Pagni (2007) revisited the same problem with a some different proofs.

Instead of approaching the problem directly, as did all the responses to Faux's article or Pagni, one could instead consider the analogous case for a (convex) quadrilateral as shown in Figure 2. Using *Sketchpad*, it will probably immediately be obvious that the two sums of the pairs of triangles at opposite vertices are not only equal, but also equal to the area of the (convex) quadrilateral. But WHY is this so?

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The following proof is remarkably simple. Since two right angles are attached to each vertex, the angle of each triangle attached at a vertex of the quadrilateral is supplementary to the corresponding interior angle of the quadrilateral. Therefore, Area $\Delta A = \frac{1}{2} ad \sin(\angle A) = \frac{1}{2} ad \sin(\angle BAD) = \text{area } \Delta BAD$, and so on. Thus, area ΔA + area ΔC = area ΔBAD + area ΔBCD = area *ABCD*. Similarly, area ΔB + area ΔD = area *ABCD*. (Note that the proof is no longer valid when the quadrilateral becomes concave, i.e. when one of the angles of the quadrilateral becomes reflexive, since the sine of that angle would become negative).

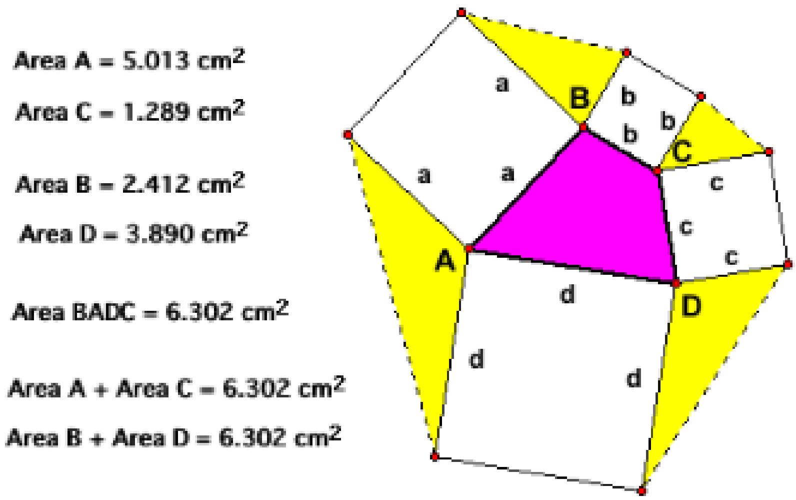

Figure 2: A generalisation to a (convex) quadrilateral.

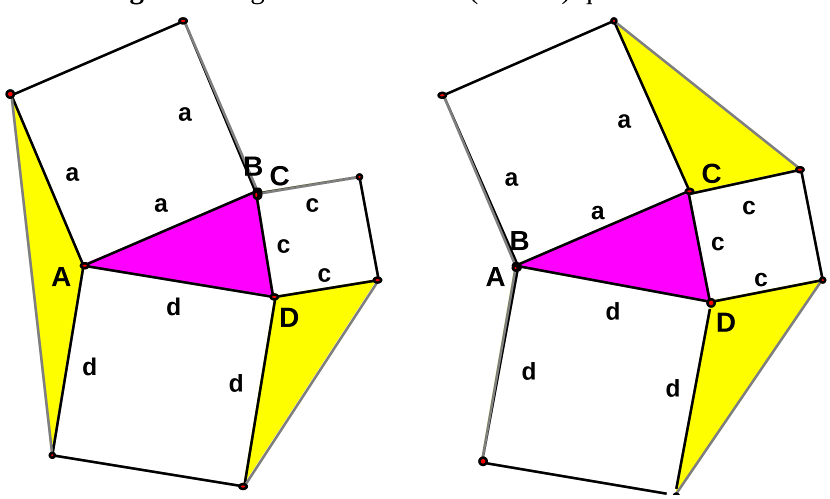

Figure 3: Specialising to a triangle.

The special case for the triangle is obtained when any two of the vertices coincide. For example, suppose *B* is dragged to coincide with *C* as shown by the first figure in Figure 3, then areas *B* and *C* become zero, and we simply have area ΔA = area ΔD = area $\Delta A(BC)D$.

But since the situation is entirely symmetrical, degenerate quadrilateral $\Delta(AB)CD$ can also be obtained by appropriately dragging vertex B to coincide with A as shown by the second figure in Figure 3, in which case area ΔC = area ΔD, and completes the proof for the special case.

Although there are many different ways of proving this result, including purely by synthetic geometry, the beauty and value of the given trigonometry proof is that it clearly explains why the result is true. This deeper insight will allow us to generalize the result further without any need for further experimentation. When this happens, proof no longer just plays the role of verification, but rather that of *a priori discovery*.

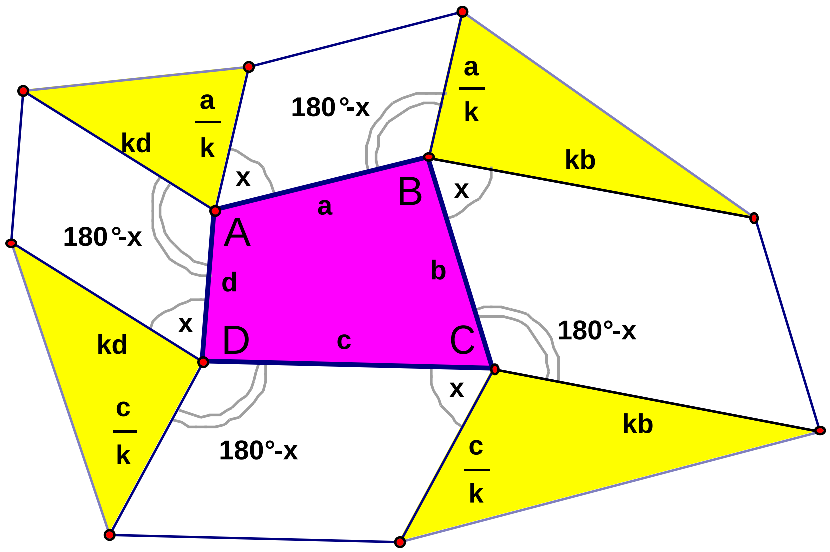

Figure 4: Generalizing to similar parallelograms.

Why is the result true? Clearly, the result depends on maintaining the supplementarity of the angle of each triangle attached at a vertex of the quadrilateral with that of the corresponding interior angle of the quadrilateral. In other words, we have to ensure that the pair of angles of the two outer quadrilaterals attached at each vertex is supplementary. Apart from squares, which other quadrilaterals have pairs of angles that are supplementary?

An immediate example that will come to mind is that of the parallelograms. Obviously, since all squares are similar, the parallelograms would also need to be similar. So how can we arrange similar parallelograms in such a way that the result is maintained? The author's recent experience has been that with some guided questions the teachers eventually come up with the arrangement of similar parallelograms shown in Figure 4 (so

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that the two relevant adjacent sides of each triangle are inversely proportional). It is easy to see that the area of the triangle attached at *A* is equal to area triangle *ADB*, since $\frac{1}{2}(kd)\left(\frac{a}{k}\right)\sin A = \frac{1}{2}ad\sin A$. Similarly, since the area of the triangle attached at *C* equals the area of triangle *BCD*, it follows that the sum of the two triangles attached at *A* and *C* is equal to area *ABCD.* Similarly, the sum of the areas the two triangles attached at *B* and *D*, are equal to area *ABCD.*

What happens if the similar parallelograms are arranged differently? Suppose, the similar parallelograms are placed as shown in Figure 5? What happens now? It is left to the reader to verify that in this case, the two sums of the areas of the pairs of triangles attached to opposite vertices are equal to $k^2 \times$ the area *ABCD.*

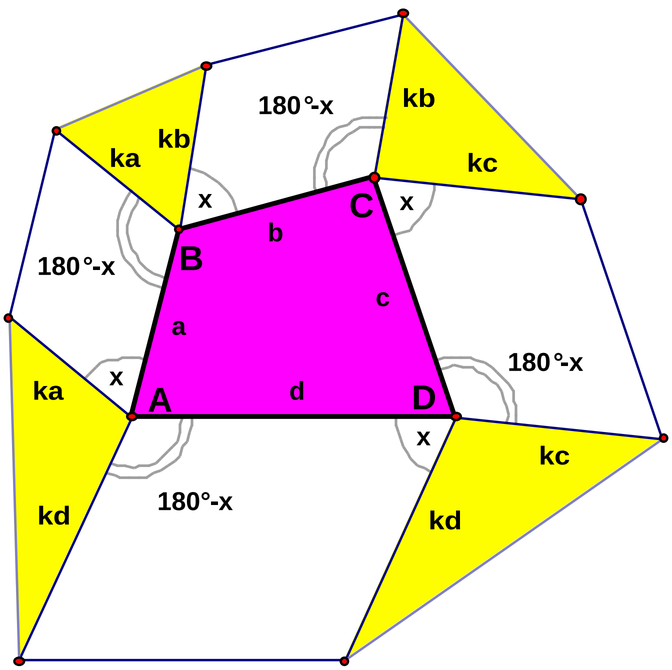

Figure 5: Another generalisation to similar parallelograms.

What other possibilities are there? Are there any other quadrilaterals that have pairs of supplementary angles that could possibly be used? The author's recent experience was that teachers soon suggested cyclic quadrilaterals, since their opposite angles are supplementary. But how should they be arranged?

With a little guidance the teachers were led to realize that the result can also be

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generalised to an arrangement of similar cyclic quadrilaterals on the sides of *ABCD* as shown in Figure 6. In this case, the sum of the areas of the two triangles attached at *A* and *C* is equal to $k_1 \times k_4 \times$ area *ABCD* while the sum of the areas of the two triangles attached at *B* and *D* is equal to $k_2 \times k_3 \times$ area *ABCD*. But from the similarity of the quadrilaterals attached to *AB* and *BC*, it follows that $\frac{k_1 a}{k_2 a} = \frac{k_3 b}{k_4 b}$ which implies that $k_1 \times k_4 = k_2 \times k_3$. Therefore, it maintains the equality of the two sums of areas of the pairs of triangles at opposite vertices.

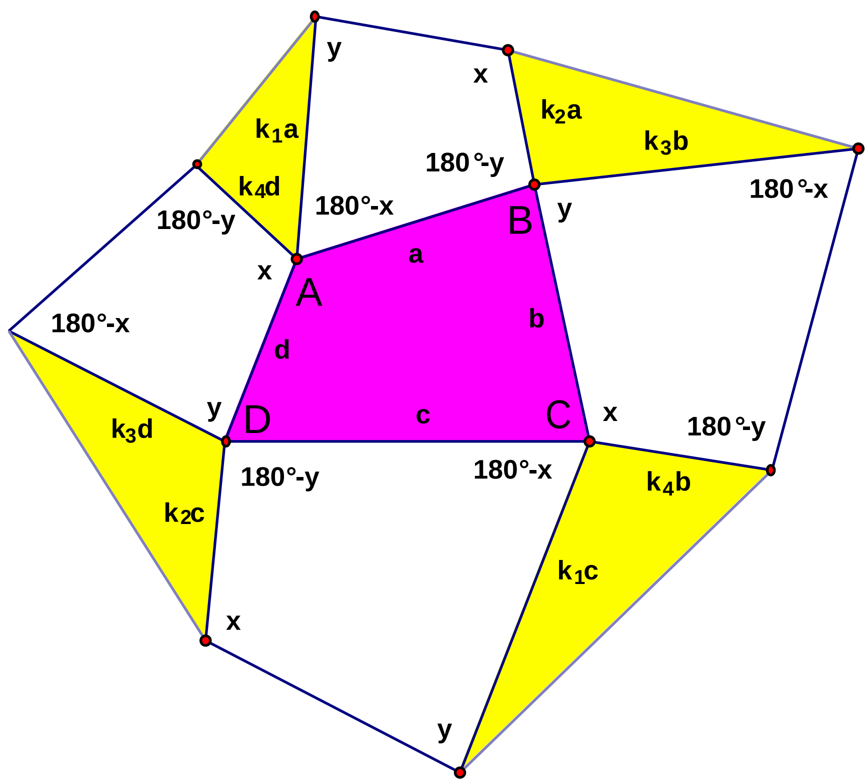

Figure 6: Generalizing to similar cyclic quadrilaterals.

Another possible avenue of further exploration that will be left to the reader is that of considering pairs of similar quadrilaterals on the opposite or adjacent sides, but not necessarily all similar to each other.

It should be noted that the generalizations presented above are not at all suggested by the purely experimental verification of the original problem. However, by proving it and identifying the fundamental property that made it true, one actually needs no further accurate experimentation. Any way, experimental exploration, if done completely blindly and randomly, would hardly be likely to deliver any profitable results.

Finally, we need to create adequate and sufficient opportunities for students at all levels, high school and university, to experience the different functions of proof, one of them being the "*discovery*" function illustrated here. Moreover, care should be taken not to present a simplistic, linear view of mathematics as always developing from experimentation to deduction, but try to illustrate a more authentic view of the dynamic interplay between experimentation and deductive thought.

Note

Dynamic Geometry (*Sketchpad 4*) sketches in zipped format (Winzip) of the results discussed here can be downloaded directly from:
http://mysite.mweb.co.za/residents/profmd/crossgeneral.zip

(If not in possession of a copy of *Sketchpad 4*, these sketches can be viewed with a free demo version of *Sketchpad 4* that can be downloaded from:
http://www.keypress.com/x17670.xml)

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