A generalization of the Reuleaux triangle

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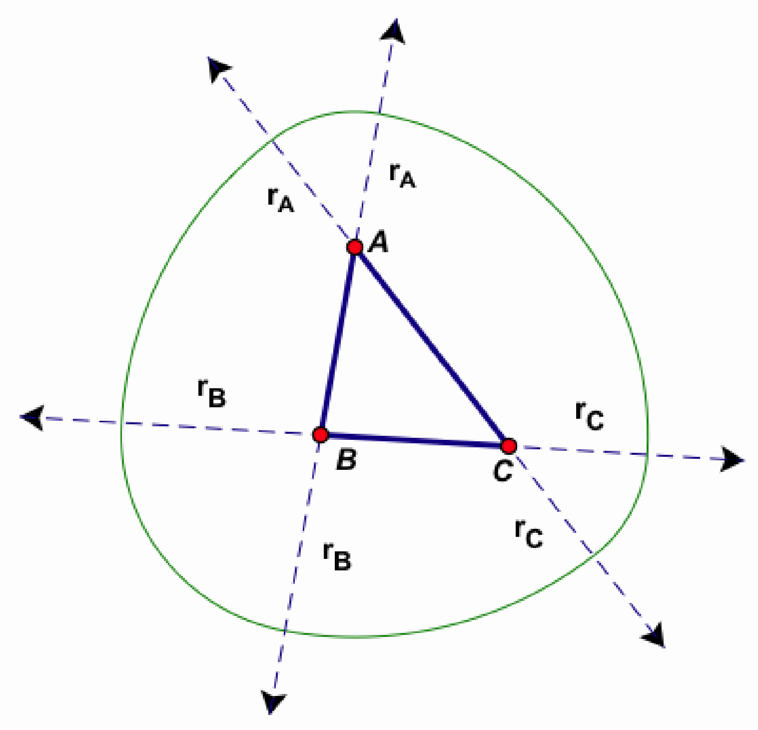
It is well known that a closed curve with constant diameter can be formed by drawing appropriate arcs centred at the vertices of an equilateral triangle. Perhaps surprisingly, the construction can be done with any (non-degenerate) triangle ABC. The sides of the triangle are extended in both directions, and, with each vertex as centre, arcs of large and radius can be drawn between the two produced sides incident at that vertex in such a way that the arcs form a smooth simple closed curve with convex interior.



Suppose without loss of generality that $\hat{A}$ is the smallest angle (not necessarily unique), so that side a is the shortest. Then draw an arc of (positive) radius r_A with centre A between sides CA produced and BA produced (that is, away from the triangle). Then with centre B draw an arc of radius $R_B = r_A + c$ between BA produced and BC produced. (This arc meets the previous one smoothly and includes the triangle.) Now continue in the same way around the triangle, alternately drawing small arcs away from the triangle and large arcs including the triangle. The radii will be respectively

$$R_B = r_A + c, \quad r_C = R_B - a, \quad R_A = r_C + b,$$

$$r_B = R_A - c, \quad R_C = r_B + a, \quad r_A = R_C - b.$$

If we eliminate the large radii, then we see that $r_C = r_A + c - a \geq r_A$, and $r_B = r_C + b - c = r_A + b - a \geq r_A$. Thus all six radii are positive, and the last equation (which is required to ensure that the curve is closed) simplifies to $r_B = r_A + b - a$, which agrees with the expression for r_B in terms of r_A.

Alternatively, from an algebraic point of view, the three equations linking r_A, r_B, r_C are not independent, since their sum reduces to 0 = 0, so they have infinitely many solutions in terms of a parameter (positive or negative), for which we chose r_A above.

Parallel tangents to the curve must touch one pair of concentric large and small arcs, so the distance between them will be $R_A + r_A$ or $R_B + r_B$ or $R_C + r_C$. From the expressions above, it is easy to see that these distances are all equal to $2r_A + b + c - a$, so the curve does have constant diameter. Geometrically speaking, the constancy is obvious, since the arcs meet smoothly, so there is no change in the tangents when passing from one pair of arcs to the adjacent pair.

It is interesting to consider other cases disallowed above. If the triangle has zero area, with $b = a + c$, say, then the four arcs centred at A and C vanish, because the angles at those vertices are zero. The angle at B is 180°, and $r_B = r_A + b - a = r_A + c = R_B$, so the two arcs centred at B form a complete circle, which of course has constant diameter.

Finally, if $r_A = 0$ in a non-degenerate triangle, then the curve is no longer smooth, because the smaller arc at A vanishes. If r_A decreases further and becomes negative, then the smaller arc moves inside the triangle. Smoothness of the curve returns, but constant diameter and convexity of the interior are lost. As r_A becomes more negative, the curve will eventually intersect itself, and will lose simplicity. However, smoothness is retained except when one of the radii is zero. Finally, when $r_A < a - b - c$, all the radii are negative. All the self-intersections untangle, and the curve returns to its original form, with each negative large or small arc playing the role previously played by its positive small or large counterpart.

A short video clip dynamically illustrating a Sketchpad construction of this generalization is available at: http://screencast.com/t/ODRjNzQ1M
(It takes a short while to load, and then press play, and be patient as it shows how the constant diameter curve changes dynamically by dragging).