The affine invariance and line symmetries of the conics

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Introduction

The following article presents some interesting affine properties of the conics that generally do not appear in standard textbooks. They were accidentally discovered while working on a series of articles on transformation geometry (De Villiers, 1989, 1990, 1992, in press). It provides a nice illustration of the power and usefulness of transformations at the undergraduate level, which in the view of the author should be compulsory for prospective senior secondary mathematics teachers.

In contrast to the isometric and similar transformation which respectively preserve the congruency and shape of transformed figures, affine transformations in general do not preserve angle size or length of line segments. Under an affine transformation the following properties of a plane geometric configuration remains *invariant* (unchanged):

- incidence of corresponding points and lines
- collinearity of corresponding points
- parallelism of corresponding lines
- ratio in which a corresponding point divides a corresponding line segment.

For example, using the special affine transforming *stretching* and *shearing* we can transform a square into a rectangle, rhombus or parallelogram, but not into a general trapezium or kite. A square is therefore not an affine invariant, but a parallelogram is, since the parallelism of opposite sides are preserved.

The affine invariance of straight lines

A standard exercise is to show that linearity is preserved under affine transformations, i.e. the invariance of straight lines. This can easily be done as follows:

Consider the general equation for a straight line, namely $y = mx + c$, and the general formulae for an affine transformation, namely:

$$x' = dx + ey + f \quad \text{and} \quad y' = gx + hy + i \qquad \text{where } dh - eg \neq 0.$$

Solving for x and y in terms of x' and y' we obtain:

$$(1)\quad x = \frac{-hx' + ey' - ei + fh}{eg - dh}$$

$$(2)\ y = \frac{gx' - dy' + di - gf}{eg - dh}$$

Substituting these into the equation for a straight line we obtain the following:

$$\frac{gx' - dy' + di - gf}{eg - dh} = m\left(\frac{-hx' + ey' - ei + fh}{eg - dh}\right) + c$$

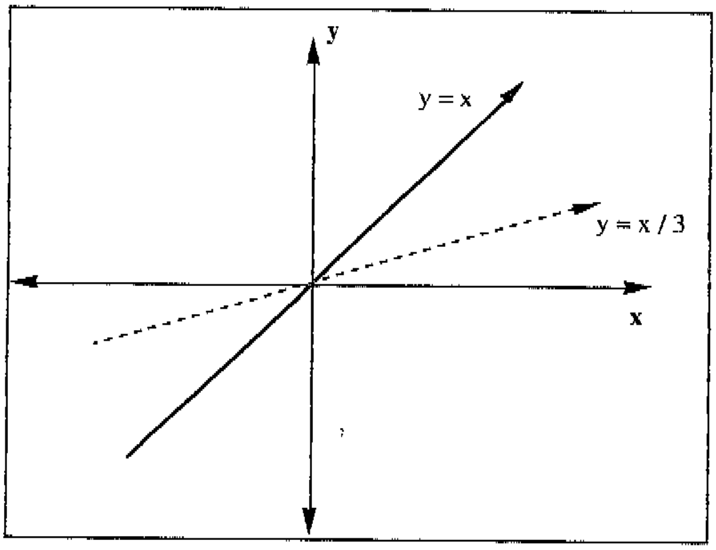
which can be simplified to the standard form:

$$y' = M'x' + C'$$

where M' and C' are expressible in terms of the constants c, d, e, f, g, h and i. This then concluded the proof of the general result.

Figure 1 shows an example of the effect of an affine transformation defined by x' = 2x + y and y' = y, on a straight line y = x

Figure 1



The affine invariance of the conics

Let us now investigate what happens to a standard parabola $y = ax^2 + bx + c$ under affine transformations. As mentioned in De Villiers (1990), stretches of $y = x^2$ in the $x-$ or $y-$ directions only produce magnifications or reductions, i.e. *similar* parabola which can be expressed in the form $y = ax^2$. But what about an affine transformation? What do we obtain?

Let's consider the transformation of $y = x^2$ under the above mentioned affine transformation:

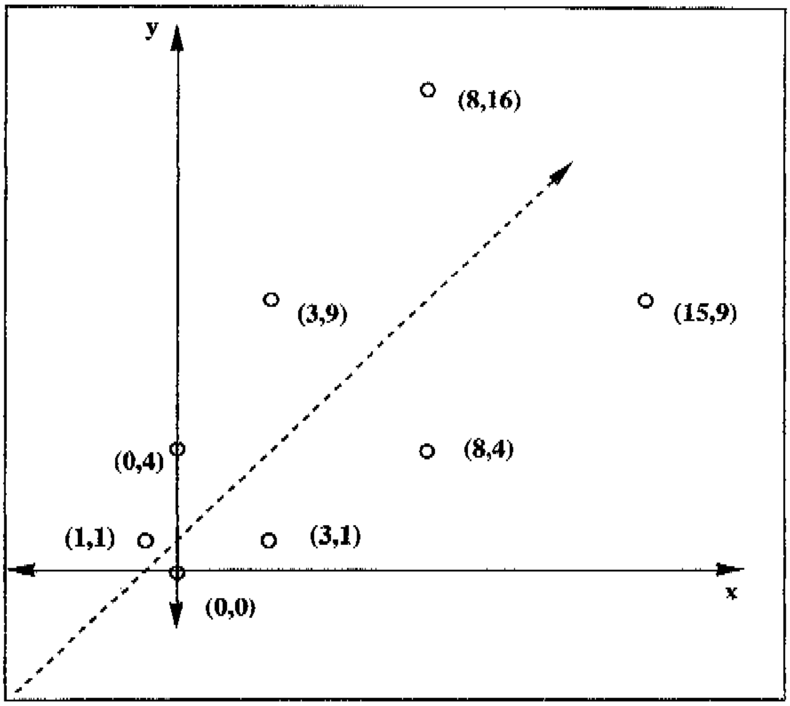
$$x' = 2x + y$$
$$and$$
$$y' = y$$

Again solving for x and y in terms of x' and y' and substituting into $y = x^2$ we obtain the transformed equation

$$x'^2 - 2x'y' + y'^2 - 4y' = 0.$$

To graph this equation we can simply consider points on the original graph and what happens to them under the given transformation. For example, the point (1,1) is mapped onto $x' = 2(1) + 1 = 3$ and $y' = 1$, therefore the point (3,1). Figure 2 shows a number of plotted points from the transformed equation which can clearly be seen to lie in the form of a parabola with an axis of symmetry $y = x + 1$. Is this always true? Will we always get a parabola?

Figure 2



Let us consider some more examples. Using the affine transformation:

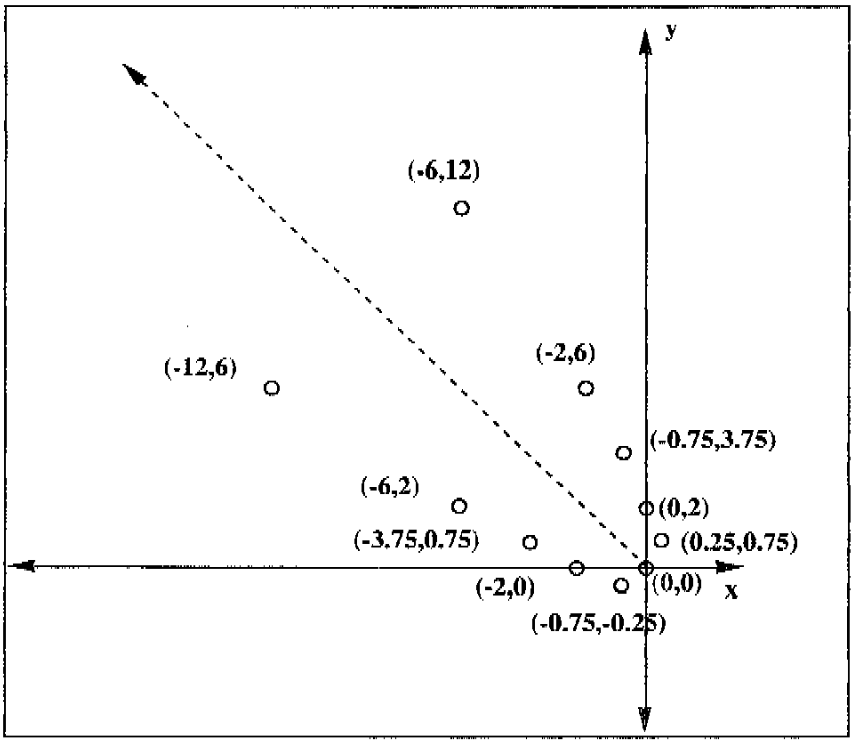
$$x' = x - y \text{ and } y' = x + y \text{ on } y = x^2$$

we obtain the transformed equation:

$$x'^2 + 2x'y' + y'^2 + 2x' - 2y' = 0$$

which, as shown in Figure 3, also lies in the form of a parabola (with an axis of symmetry $y = -x$).

Figure 3



Similarly, if we use:

$$x' = -2x + \frac{y}{2} + 1$$

and

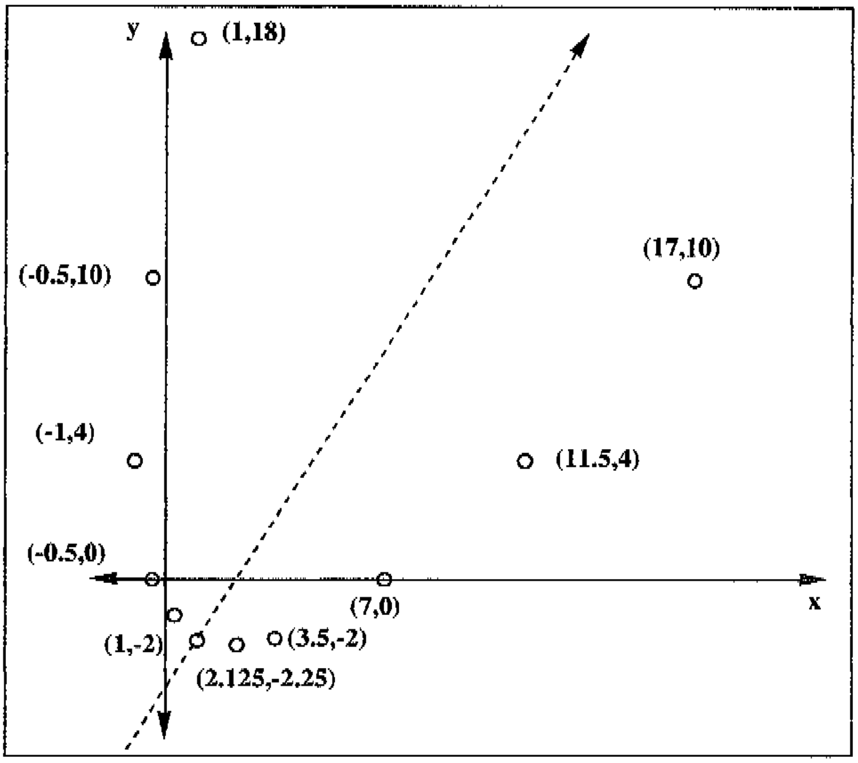
$$y' = x + y - 2$$

on $y = x^2$ we obtain :

$$4x'^2 - 4x'y' + y'^2 - 26x' - 12y' - 14 = 0$$

producing the parabolic curve shown in Figure 4. Note that the axis of symmetry cuts the x-axis at (2,0) and passes through (1,-2), implying that its equation is $y = 2x - 4$.

Figure 4



How can we prove these observations, which suggest that any affine transformation of

$$y = ax^2 + bx + c$$

always produces a parabola? For this purpose it is necessary to consider the general representation of a plane conic:

$$px^2 + 2qxy + ry^2 + 2sx + 2ty + u = 0$$

This equation represents a parabola if $q^2 - pr = 0$, an ellipse if $q^2 - pr < 0$ and a hyperbola if $q^2 - pr > 0$. Note that it represents a circle if $p = r$ and $q = 0$, and a rectangular hyperbola if $p + r = 0$. (A proof is given in Fishback, 1969, pp. 215-217).

Substituting equations (1) and (2) (i.e. the solutions of x and y in terms of x' and y' from the general affine equations) into the general equation for a parabola $y = ax^2 + bx + c$, we obtain the transformed equation:

$$ah^2x'^2 - 2aehx'y' + ae^2y'^2 + 2S'x' + 2T'y' + U' = 0$$

Since $(-aeh)^2 - (ah^2)(ae^2) = 0$ this equation clearly represents a parabola. However, this result suggests the more general result that any parabola:

$$px^2 + 2qxy + ry^2 + 2sx + 2ty + u = 0$$

with $g^2 - pr = 0$ is an affine invariant. In this case we obtain the transformed equation:

$$(ph^2 - 2qhg + rg^2)x'^2 + 2(qeg + qhd - hep - gdr)x'y' + (pe^2 - 2qed + rd^2)y'^2 + 2S'x' + 2T'y' + U' = 0$$

The discriminant condition is now given by:

$$(qeg + qhd - hep - gdr)^2 - (ph^2 - 2qhg + rg^2)(pe^2 - 2qed + rd^2)$$

which simplifies to:

$$q^2e^2g^2 + q^2h^2d^2 + 2hepgdr - prh^2d^2 - 2q^2hged + rpg^2e^2$$

Since $q^2 = pr$ in the case of a parabola, we can replace pr by q^2 in the 3rd, 4th and 6th term upon which this expression reduces to 0, thus proving the affine invariance of a parabola.

Similarly, if we had an ellipse to start with, we can replace pr by $q^2 + k$ where $k > 0$ so that the expression reduces to $-k(hd - ge)^2$. Since this value is always negative, the transformed equation is that of an ellipse and proves the affine invariance of an ellipse. In the same way we can prove the affine invariance of a hyperbola, but that is left as an exercise to the reader.

The affine equivalence of the conics

A standard exercise in transformation geometry is to show that any conic can be reduced to a corresponding canonical (simpler) form by appropriate transformations. For example, any parabola can be reduced to $x^2 + y = 0$, any ellipse (excluding here imaginary cases) to $x^2 + y^2 = 1$ and any hyperbola to $x^2 - y^2 = 1$.

This can be done in a variety of ways. Fishback (1969) for example uses a combination of affine and isometric transformation to illustrate this. Pettofrezzo (1966) on the other hand uses only isometric transformations, namely, a rotation, a translation and a reflection around $y = x$ if necessary, to reduce a general parabola to the form:

$$ax^2 + y + c = 0,$$

a general ellipse to the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and a general hyperbola to:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The latter forms can however be further reduced by appropriate transformations. For example, the given parabola can easily be reduced to the first form above by using a translation to eliminate c followed by a magnification of a from the origin (see De Villiers, 1990). To further reduce the given ellipse and hyperbola to the first forms above, an affine stretch of $x' = x$ and $y' = ay$,

followed by a magnification of $x' = \frac{x}{a}$ and $y' = \frac{y}{a}$ is required.

Since each conic can be reduced to canonical form by means of a suitable combination of affine transformations (including isometries and similarities), it is clearly also possible to map any two cases of a particular conic onto each other with a suitable combination of affine transformations, (e.g. if necessary we can reduce the one to canonical form, and then simply use the inverse transformations by which the other would be reduced to canonical form, to map it onto the other). We can therefore say that each individual conic is affine equivalent. The reduction method described earlier for parabola also implies that any parabola can be mapped onto any other by only using a combination of the isometrics and similarities, and therefore the stronger result that all parabola are similar. Note that ellipses and hyperbolas are not individually similar, since they require affine transformation to be reduced to canonical form. Also note that it is not possible to affinely transform one conic, e.g. a parabola, to another, say an ellipse. For that purpose, projective transformations are required and it can be proved that all conics are projective equivalent.

One should be careful not to confuse the different concepts of affine invariance and affine equivalence. The fact that a particular set of figures is invariant under certain transformations does not necessarily mean that all the figures from that set are equivalent under those transformations. For example, every affine transformation sends a convex quadrilateral to another convex quadrilateral, so convex quadrilateral is an affine invariant. It is not true, however, that any two convex quadrilaterals are affine equivalent. For example, a parallelogram cannot be affinely mapped into a quadrilateral with only one pair of opposite sides equal. Conversely, equivalence does not necessarily imply invariance. For example, all rectangles are affine equivalent, i.e. can be mapped exactly onto each other by suitable stretches, but a rectangle is not an affine invariant, since a shear for example, can map it onto a parallelogram.

The line symmetries of the conics

How can we find the line symmetries of parabolas, ellipses and hyperbolas in general? Is it possible to derive general formulae?

Let us first consider a general parabola defined by :

$$px^2 + 2qxy + ry^2 + 2sx + 2ty + u = 0$$
$$and$$
$$q^2 = pr$$

After a rotation of the graph around the origin given by:

$$x' = x\cos\theta - y\sin\theta$$
$$and$$
$$y' = x\sin\theta + y\cos\theta$$
$$where$$
$$\theta = \frac{1}{2}\arctan\left[\frac{2q}{(r-p)}\right]$$

we obtain one of the following two transformed equations (under the conditions shown, which follow easily from considering a couple of special cases):

$$(3)\quad \begin{cases} (p+r)x'^2 + 2(s\cos\theta - t\sin\theta)x' + 2(s\sin\theta + t\cos\theta)y' + u = 0 \\ if \\ |r| < |p| \\ or \\ r = p < 0 \end{cases}$$

$$(4)\quad \begin{cases} (p+r)y'^2 + 2(s\cos\theta - t\sin\theta)x' + 2(s\sin\theta + t\cos\theta)y' + u = 0 \\ if \\ |r| > |p| \\ or \\ r = p > 0 \end{cases}$$

Now equations (3) and (4) are simply standard parabolas with vertical and horizontal axes of symmetry given by the respective equations:

$$x' = -\frac{s\cos\theta - t\sin\theta}{p+r}$$

$$y' = -\frac{s\sin\theta + t\cos\theta}{p+r}$$

Since a rotation is an isometry, the transformed parabolas with their lines of symmetry are congruent to the original. In order to obtain general equations for the lines of symmetry for parabolas in general, we need therefore only rotate the above lines of vertical or horizontal symmetry back to their original positions by using the following transformations:

$$\begin{aligned} &x = \cos(-\theta)x' - \sin(-\theta)y' \\ &\textit{and} \\ &y = \sin(-\theta)x' + \cos(-\theta)y'. \end{aligned}$$

Solving for x' and y' in terms of x and y and substituting into the above equations, we obtain:

$$\cos(-\theta)x + \sin(-\theta)y = -\frac{s\cos\theta - t\sin\theta}{p+r}$$

$$-\sin(-\theta)x + \cos(-\theta)y = -\frac{s\sin\theta + t\cos\theta}{p+r}$$

which reduce to the following general formulae for the line of symmetry of a parabola:

$$(5) \qquad \begin{cases} y = x\cot\theta + \dfrac{s\cot\theta - t}{p+r} \\ \quad\quad if \\ \quad\quad |r| < |p| \\ \quad\quad or \\ \quad\quad r = p < 0 \end{cases}$$

$$(6)\quad \begin{cases} y = -x\tan\theta - \dfrac{s\tan\theta + t}{p+r} \\ if \\ |r| > |p| \\ or \\ r = p > 0 \end{cases}$$

It should be noted that:

$$\theta = \frac{1}{2}\arctan\left[\frac{2q}{(r-p)}\right]$$

is the angle between the line of symmetry and the y- or x-axes respectively, dependent on the conditions given in formulae (5) and (6). Furthermore, according to standard convention $\theta > 0$ implies clockwise rotation. Also note the restriction that $\theta \neq 0°$ in formula (5), since we divided by $\sin\theta$ to obtain it. When $\theta \neq 0$, q = 0 and the axis of symmetry is them simply $x = \frac{s}{p}$. No such restriction is necessary in formula (6) since the domain of θ is $-45° \leq \theta \leq 45°$.

Let's now look at some examples. Firstly consider the parabola shown in Figure 4, namely :

$$4x^2 - 4xy + y^2 - 26x - 12y - 14 = 0$$

Here we have $\theta = 0.5\arctan\left[\frac{2(-2)}{(1-3)}\right] = 26.565°$. Since $|r| < |p|$ we determine $\cot\theta = 2$ and substitute it with p, r, s and t into formula (5) above to obtain $y = 2x - 4$.

Similarly for the parabola:

$$x^2 + 2xy + y^2 + 2x - 2y = 0$$

shown in Figure 3, we obtain $\theta = 45°$ and since $r = p > 0$ we substitute into formula (6) to obtain $y = -x$ as the axis of symmetry. However, for the parabola:

$$x^2 - 2xy + y^2 - 4y = 0$$

shown in Figure 2 we obtain $\theta = -45°$ and since $r = p > 0$ we also substitute into formula (6) to obtain $y = x + 1$ as the axis of symmetry. It is now left as an exercise for the reader to show that is $y = -\frac{1}{2}x - 2$ is the axis of symmetry of the parabola:

$$x^2 + 4xy + 4y^2 - 12x + 26y - 14 = 0$$

We can now easily draw a rough graph of the latter parabola by considering the direction in which its 'arms' are lying. From formula (4) we can see that a general parabola with:

$$|r| > |p|$$

would lie in the positive or negative x-direction respectively when the quotient:

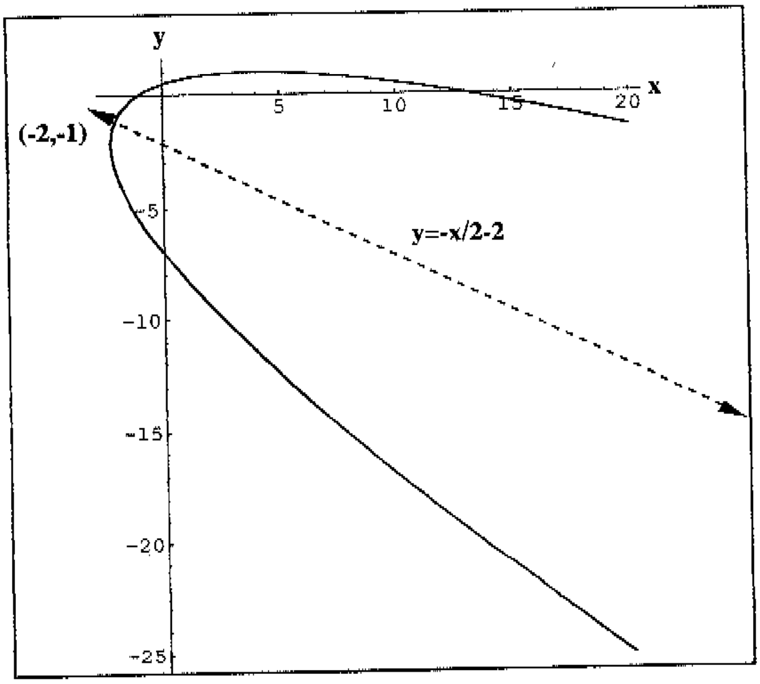
$$-\frac{(p+r)}{(s\cos\theta - t\sin\theta)}$$

is positive or negative. Since $\theta > 0$, $s < 0$ and $t > 0$ it follows that this quotient is positive and we can draw the rough graph of:

$$x^2 + 4xy + 4y^2 - 12x + 26y - 14 = 0$$

as shown in Figure 5.

Figure 5



In the same manner we can deduce that in general a parabola with $|r| < |p|$ or $r = p < 0$ would lie in the positive or negative y-direction respectively when the quotient

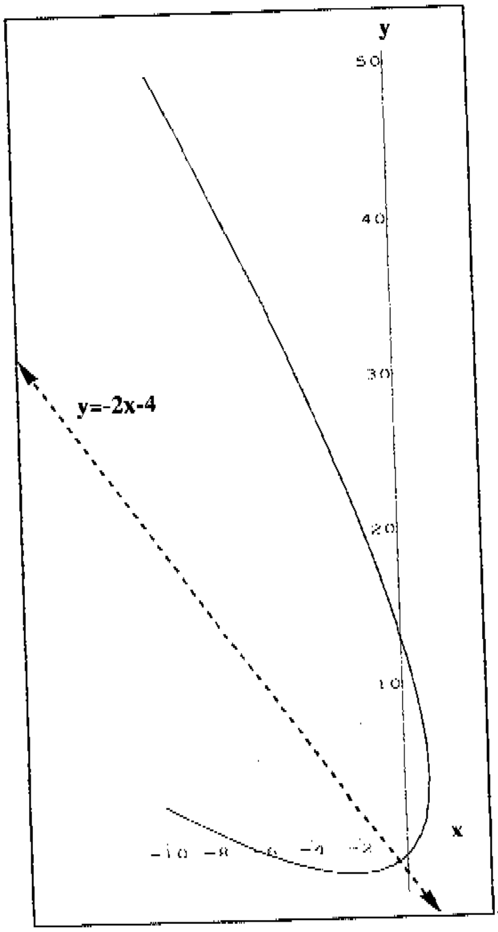
$$-\frac{(p+r)}{(s\sin\theta - t\cos\theta)}$$

is positive or negative. For example, since the axis of symmetry of the parabola:

$$4x^2 + 4xy + y^2 + 26x - 12y - 14 = 0$$

is $y = -2x - 4$ and its aforementioned quotient is positive, we can easily draw its rough graph as shown in Figure 6.

Figure 6



Of course, to determine the x- and y-intercepts, we respectively set $y = 0$ and $x = 0$, and then correspondingly solve for x and y.

In order to derive equations for the coordinates of the turning point of a parabola in general, we first find the coordinates of the turning points of parabolas (3) and (4). For example:

$$x'_t = \frac{-\alpha}{p+r} \quad \text{and} \quad y'_t = \frac{\alpha^2 - u(p+r)}{2\beta(p+r)} \qquad |r| < |p| \quad \text{or} \quad r = p < 0$$

$$x'_t = \frac{\beta^2 - u(p+r)}{2\alpha(p+r)} \quad \text{and} \quad y'_t = \frac{-\beta}{p+r} \qquad |r| > |p| \quad \text{or} \quad r = p > 0$$

where:

$$\alpha = s\cos\theta - t\sin\theta \quad \text{and} \quad \beta = s\sin\theta + t\cos\theta$$

Since x'_t and y'_t are the coordinates of a fixed point we directly use the transformations

$$x'_t = \cos(-\theta)x'_t - \sin(-\theta)y'_t \quad \text{and} \quad y'_t = \sin(-\theta)x'_t + \cos(-\theta)y'_t$$

to obtain the following formulae for the coordinates of the turning points of a parabola in general:

$$(7) \quad \begin{cases} x'_t = \dfrac{-2ab\cos\theta + [\alpha^2 - u(p+r)]\sin\theta}{2\beta(p+r)} \\ y'_t = \dfrac{2\alpha\beta\sin\theta + [\alpha^2 - u(p+r)]\cos\theta}{2\beta(p+r)} \\ |r| < |p| \\ \text{or} \\ r = p < 0 \end{cases}$$

$$(8)\quad \begin{cases} x'_t = \dfrac{-2\alpha\beta\sin\theta + [\beta^2 - u(p+r)]\cos\theta}{2\alpha(p+r)} \\ y'_t = \dfrac{-2\alpha\beta\cos\theta + [\beta^2 - u(p+r)]\sin\theta}{2\alpha(p+r)} \\ |r| > |p| \\ or \\ r = p > 0 \end{cases}$$

For example, for the parabola shown Figure 6 we had θ = -26.565° ... which gives us α = 8.944..., β = -11.180..., $\alpha\beta$ = -100 and α^2 = 80. By substitution into formula (7) we obtain the coordinates of the turning point as follows:

$$x_t = \frac{-2(-100)(0.894\ldots) + 150(-0.447\ldots)}{10(-11.180\ldots)} = \frac{111.803\ldots}{-111.803\ldots} = 1$$

$$y_t = \frac{-2(-100)(-0.447\ldots) + 150(0.894\ldots)}{10(-11.180\ldots)} = \frac{223.606\ldots}{-111.803\ldots} = 2$$

It is now left as an exercise for the reader to verify that the turning points of the parabolas shown in Figures 2 and 5 are respectively (-0.75,0.25), (0,0), (1,-2), and (-2,-1).

Let us now consider the axes of symmetry of ellipses and hyperbolas in general. After the rotation of a general ellipse or hyperbola around the origin through the angle θ, we obtain the following transformed equation:

$$(9)\ \lambda_1 x'^2 + \lambda_2 y'^2 + 2\alpha x' + 2\beta y' + u = 0$$

where a and b are defined as in equations (7) and (8), and l$_1$ and l$_2$ are the solutions of the characteristic equation :

$$\lambda^2 - (p+r)\lambda + pr - q^2 = 0$$

Now equation (9) is simply a standard (rectified) ellipse or hyperbola with two axes of symmetry $x' = \dfrac{\alpha}{\lambda_1}$ and $y' = \dfrac{\beta}{\lambda_2}$. By again rotating these lines back to their original positions we respectively obtain the following equations for the axes of symmetry in general:

$$(10)\qquad y = x\cot\theta + \frac{s\cot\theta - t}{\lambda_1} \ldots.\ \theta \neq 0^o$$

$$(11) \qquad y = -x\tan\theta + \frac{s\tan\theta + t}{\lambda_2}$$

Note that in the above formulae, if:

$$|r| < |p|$$
$$or$$
$$r = p < 0$$

we have:

$$|\lambda_1| > |\lambda_2|$$

but if:

$$|r| > |p|$$
$$or$$
$$r = p < 0$$

we have:

$$|\lambda_1| < |\lambda_2|.$$

Furthermore, in contrast to the ellipse or parabola, the hyperbola can have the additional condition that $r = -p$, in which case $\lambda_1 = -\lambda_2$. In that case, if $r \geq 0$ we have $\lambda_1 < \lambda_2$, but if $r < 0$ we have $\lambda_1 > \lambda_2$.

Let's now consider some examples. For the ellipse:

$$52x^2 - 72xy + 73y^2 + 400x - 950y + 2725 = 0$$

we have $\theta = -36.869°$ ... and $\lambda^2 - 125\lambda + 2500 = 0$.

Since:

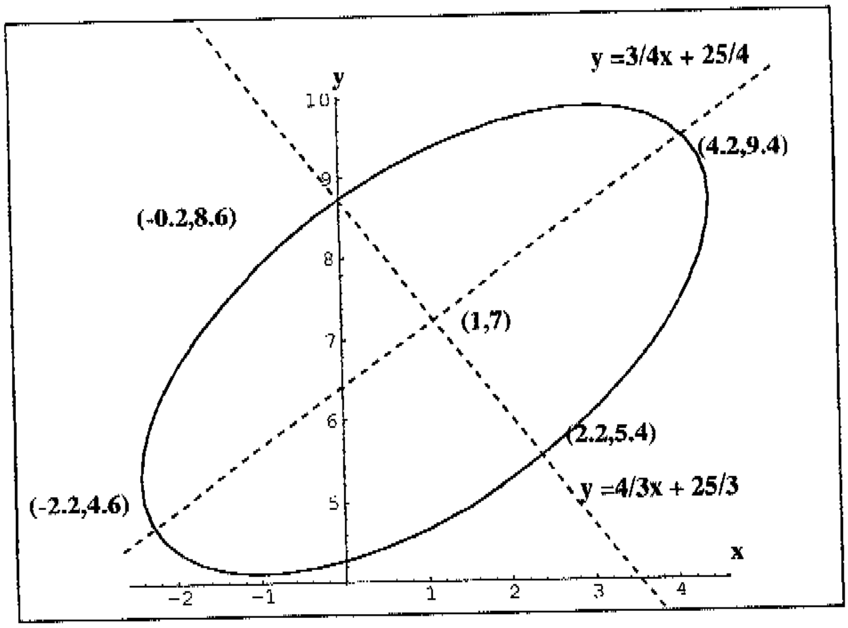
$$|r| > |p|,$$

it follows that $\lambda_1 = 25$ and $\lambda_2 = 100$ and by substitution into equations (10) and (11), we obtain the respective axes of symmetry as:

$$y = -\frac{4}{3}x + \frac{25}{3} \quad \text{and} \quad y = \frac{3}{4}x + \frac{25}{4}$$

Solving these two simultaneous equations gives us the centre of the ellipse at (1,7). Of course, by substituting these two equations into the equation of the ellipse, we can obtain the coordinates of the intersections between the axes of symmetry and the ellipse in Figure 7.

Figure 7



For the hyperbola:

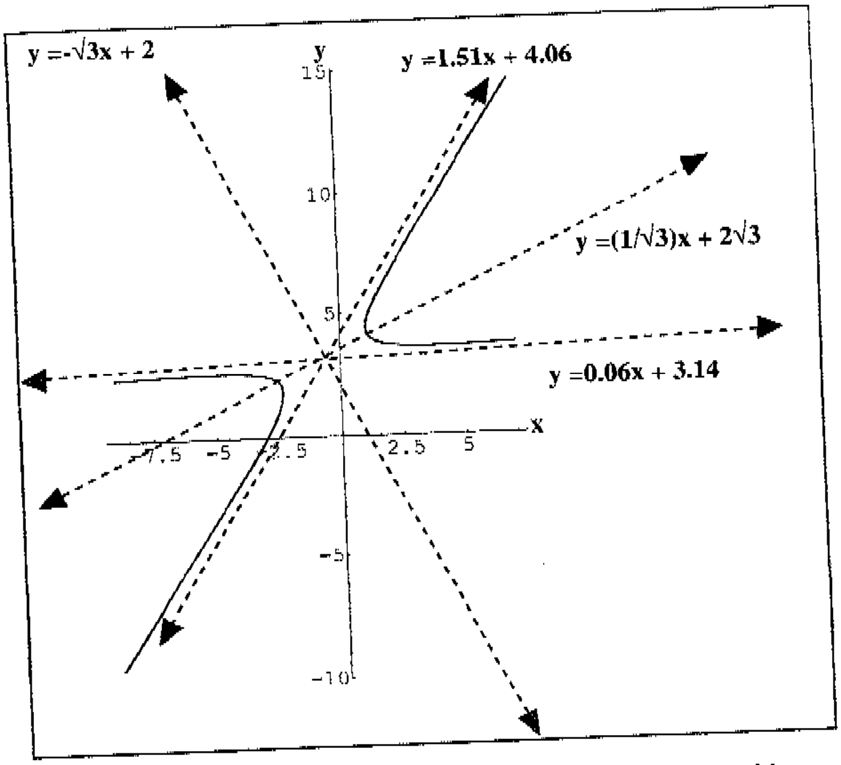
$$x^2 - 10\sqrt{3}xy + 11y^2 + 4(\sqrt{3}+12)x - 4(12\sqrt{3}-1)y + 156 = 0$$

we have $\theta = -30°$ and $\lambda^2 - 12\lambda - 64 = 0$. Since $|r| > |p|$ we must have $|\lambda_1| < |\lambda_2|$ and therefore $\lambda_1 = -4$ and $\lambda_2 = 16$. This then gives us the axes of symmetry as:

$$y = -\sqrt{3}x + 2 \text{ and } y = \frac{1}{\sqrt{3}}x + 2\sqrt{3}.$$

By substitution as before we can now obtain the graph shown in Figure 8 with coordinates rounded off to the first decimal.

Figure 8



General equations for the asymptotes of a hyperbola can also be found by considering its *rectified* position and utilising the previously demonstrated methods, but that is left as an exercise to the reader. For the hyperbola shown in Figure 8, the asymptotes are:

$$y = 1.51x - 4.06 \text{ and } y = 0.06x + 3.14 \text{ (coefficients rounded off to 2 decimals).}$$

For the hyperbola:

$$xy + \sqrt{2}x - 2\sqrt{2}y - 6 = 0$$

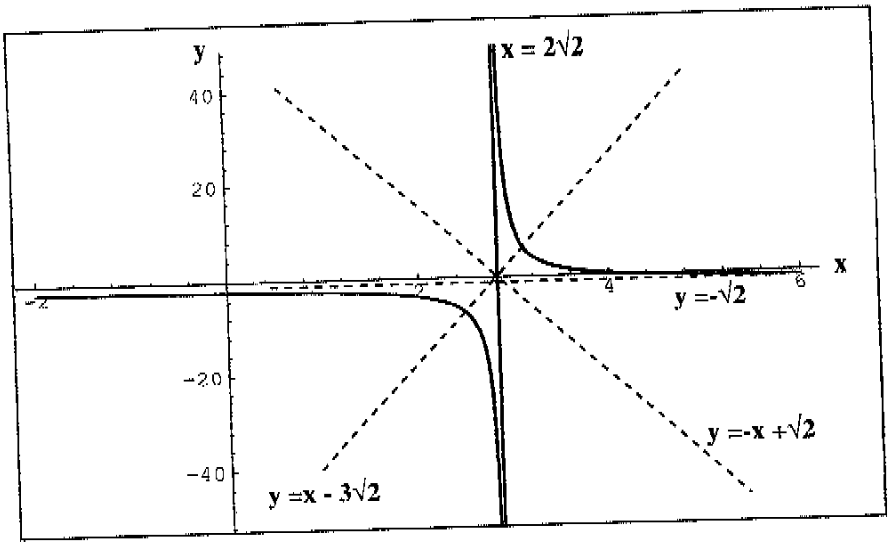
we have $\theta = 45°$ and $\lambda^2 - \frac{1}{4} = 0$. Since r = 0 we must have $\lambda_1 < \lambda_2$ and therefore $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = \frac{1}{2}$. This then gives us the axes of symmetry as:

$$y = x - 3\sqrt{2} \text{ and } y = -x + \sqrt{2}$$

By substituting as before, we obtain the graph shown in Figure 9 with asymptotes:

$$x = 2\sqrt{2} \text{ and } y = -\sqrt{2}$$

Figure 9



Concluding remarks

Hopefully this article has not only succeeded in showing the power of transformations, but also shed a bit of light on the way in which new mathematics is often discovered or created. For example, from the inductive generalisation from a couple of special cases it was firstly hypothesised that the standard parabola was an affine invariant. The deductive explanation of this conjecture however immediately led to the further generalisation that the general conics were affine invariants. At the same time this discovery prompted an investigation of equations for their line symmetries. Although the latter investigation was mainly pursued in a deductive fashion, the consideration of special cases was nevertheless necessary to identify the various discriminating conditions. We therefore clearly see in this example how deductive and quasi-empirical thought complement each other in mathematics.

Furthermore, it should be pointed out that although the conceptual content is relatively elementary, the required technical proficiency is much more complex. In other words, the greatest drawback and potential stumbling block in the formal treatment of transformation geometry does not lie so much in *relational* understanding, but in the *instrumental* mastery of it. Consult Skemp (1971) for more information about these two dimensions of mathematics teaching.

On the other hand, new powerful symbolic manipulators for personal computers such as *Mathematica*, *Theorist* or *Derive* can easily handle the instrumental manipulation of algebraic equations such as those in this article. In other words, it can free us from rote, boring and/or complex manipulations, as well as the danger of making mistakes, so that we can focus more on conceptual aspects. In fact, the author is disappointed that he did not have such a program available at the time he started this article, as it would have saved him a lot of unnecessary effort and time.

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