An offprint from

The Mathematical Gazette

The Presidential Address

Volume 97
Number 540
November 2013

www.m-a.org.uk

THE MATHEMATICAL ASSOCIATION £33.00

3D Generalisations of Viviani's theorem

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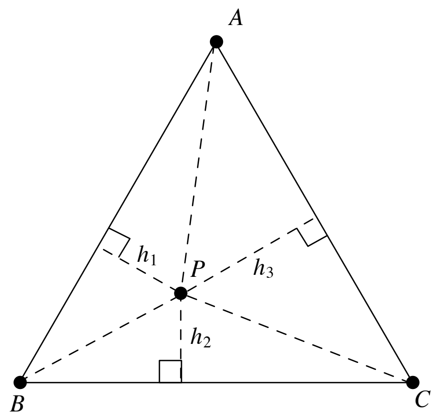
> "The ambitious learner should carefully study a new fact; he should turn it over and over, consider it under various aspects, scrutinize it from all sides … Moreover, he should try to expand and enlarge any newly acquired knowledge by application, generalization, specialization, analogy, and in all other ways." [1]

Vincenzo Viviani, a 17th century mathematician, proved that the sum of the (perpendicular) distances from a point within an equilateral triangle to its sides is constant as shown in Figure 1. The theorem, named after him, generalises to polygons that are equilateral or equi-angled, or to $2n$-gons with opposite sides parallel. Viviani's theorem is easily proved by summing the areas of triangles *APB*, *BPC* and *CPA*, equating to the area of the triangle *ABC*, and then simplifying to obtain $h_1 + h_2 + h_3 = H$, where H is the altitude of *ABC*.



$h_1 + h_2 + h_3 =$ constant

FIGURE 1: Viviani's theorem

A regular classroom investigation that I engage my students in after we've explored not only Viviani's theorem, but its extensions in 2D, is to challenge them to explore the generalisation of the theorem to 3D by considering what the analogies in 3D are for 2D concepts like triangle, side and area. It helps to reflect carefully on the 2D proofs in the 'looking back' style of Polya, considering why these results are true in 2D, and then generalising to 3D. This provides further illustrative examples of what has been called the *discovery* function in [2], that is, where a proof, by revealing the underlying *explanatory* property, can allow one to generalise further.

Since the tetrahedron and its faces are the 3D analogues of the triangle and its sides, the following generalisation as pointed out in [3], follows naturally:

Theorem 1: The sum of the distances from a point P to the faces of a tetrahedron $ABCD$ with faces of equal area, is constant (see Figure 2).

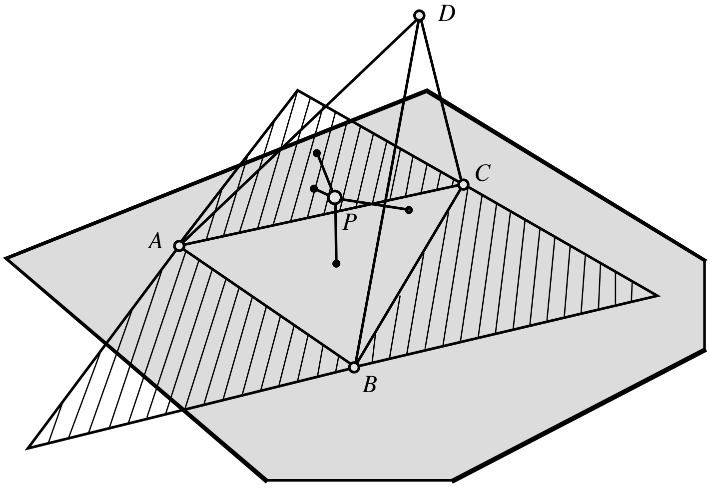


FIGURE 2: Disphenoid illustrating Viviani in 3D

As shown in [4] and [5] such a tetrahedron has congruent (acute-angled) faces and is called a *disphenoid*. Also shown in Figure 2 is a net that folds up to produce a disphenoid. It is obtained by constructing the midpoints of the sides of any acute-angled triangle to form four congruent triangles. *Cabri 3D* is particularly useful software to create a dynamic construction to illustrate the result. The proof now follows similarly to the 2D case.

Proof: Since a point P inside a disphenoid divides it into four smaller tetrahedra so that if A represents the area of each face and h_i the four heights, then $\frac{1}{3}A(h_1 + h_2 + h_3 + h_4) = V \Leftrightarrow h_1 + h_2 + h_3 + h_4 = 3V/A$, where V represents the volume of the disphenoid. Therefore, since V and A are constant, the sum of the distances from P to the faces is constant.

(Note: An interesting property of the disphenoid is also apparent from the above, namely, that the four heights (altitudes) of the disphenoid are equal since each face has the same area, and the volume calculated from each face has to be the same.)

Using the same argument as above, Theorem 1 generalises to any polyhedron with faces of equal area, and therefore not only includes the other four regular polyhedra, but also irregular ones like the hexagonal bipyramid (see Figure 3).

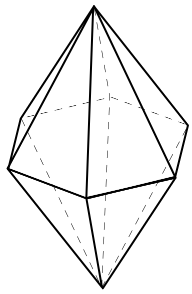


FIGURE 3: Hexagonal bipyramid

Converse of Theorem 1: If the sum of the distances from a point P to the faces of a tetrahedron is constant, then the tetrahedron has faces of equal area (i.e. is a disphenoid).

Proof: Since the sum of distances to the faces is constant, moving point P to each of the vertices shows that the tetrahedon has four equal heights (altitudes) from each of the vertices. But since the volume is constant, irrespective of whichever face it is determined from, all the faces have the same area.

Theorem 2: If a prism is created by the orthogonal translation in space of an equilateral polygon, equi-angled polygon or a $2n$-gon with opposite sides parallel, then the sum of the distances from a point P to the faces of the prism, is constant (see Figure 4).

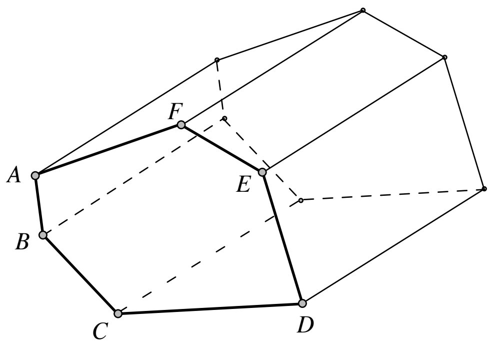


FIGURE 4: A prism created with *ABCDEF*, a hexagon with opposite sides parallel

Proof: In the case of prisms formed by the orthogonal translation of these polygons, the distances from P to the side faces formed by the translation all lie in the same plane parallel to the original polygon, and the intersection of this plane with the prism produces a polygon congruent to the original. Moreover, the translated polygon and its image are parallel; hence the result.

5. T. Andreescu and R. Gelca, *Mathematical Olympiad challenges* (2nd edn.), Birkhauser (2009) pp. 30-31.
6. Wikipedia: Viviani's theorem. Accessed on 26 August 2012 at: http://en.wikipedia.org/wiki/Viviani's_theorem
7. M. T. Edwards and A. Quesada, Dueling (Dualing) solids: Enhancing student and teacher geometrical understanding with Cabri 3D. Paper presented at the Nineteenth Annual International Conference on Technology in Collegiate Mathematics, Boston, Massachusetts, February 15-18, 2007. Available at: http://archives.math.utk.edu/ICTCM/VOL19/S073/paper.pdf

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Dynamic geometry sketches with Cabri 3D illustrating these results are available at: http://frink.machighway.com/~dynamicm/disphenoid-viviani.html (Currently the Java applet is unfortunately only working with Windows computers and Macs running older operating systems and older browsers (not on OS 10.7 or higher and Java browsers newer than March 2013).