Learning and Teaching Mathematics

Chief Editor:
Duncan Samson  Rhodes

Editors:
Marcus Bizony  Bishops
Lindiwe Tshabalala  Gauteng

Online Editor:
Alwyn Olivier  Stellenbosch

Editorial Board:
Lynn Bowie  Gauteng
Biddy Cameron  W Cape
Michael de Villiers  KZN
Busi Goba  KZN
Mellony Graven  E Cape
Susan Koen  N Cape
Agatha Lebethe  W Cape
Sharon McAuliffe  W Cape
Tulsi Morar  E Cape
Themba Mthetwa  E Cape
Elija Nkosi  Mpumalanga
Stephen Sproule  Mpumalanga
Hamsa Venkatakrishnan  Gauteng

Learning and Teaching Mathematics is a journal of the Association for Mathematics Education of South Africa (AMESA). This journal is aimed at mathematics teachers at primary and secondary school level and it provides a medium for stimulating and challenging ideas, offering innovation and practice in all aspects of mathematics teaching and learning in school. Learning and Teaching Mathematics aims to inform, enlighten, stimulate, challenge, entertain and encourage mathematics educators. Its emphasis is on addressing the challenges that arise in the mathematics classroom. It presents articles that describe or discuss mathematics teaching and learning through the eyes of practising teachers and learners. While this journal ‘listens’ to research and considers it in the activities, lesson ideas, and teaching strategies that it publishes, it is not a research publication.

Articles submitted will be reviewed by the editors and members of the Editorial Board. The Board will ensure that the papers make a contribution to our understanding of mathematics learning and teaching, that the mathematics presented is correct, and that the language and layout used is user friendly. Support will be provided by the editors to contributors in relation to meeting the above requirements. The main criterion of acceptance is that the article should make a contribution to the improvement of school mathematics teaching and learning. See the inner back cover for more information on the submission of materials and articles for publication.
TABLE OF CONTENTS

From the Editors 2

Interview with a Mathematics Doodler – Dr Sizwe Mabizela, Deputy Vice Chancellor, Rhodes University 3
Mellony Graven & Marc Schäfer

Warm-up Activities for Grade 7s 6
Mark Rushby

“Completing the Square” – A Conceptual Approach 8
Duncan Samson

ANAs: Possibilities and Constraints for Mathematical Learning 12
Mellony Graven & Hamsa Venkatakrishnan

Equality is Not Always ‘Best’ 17
Michael de Villiers

Number Line Image Generator – A Website Review 22
Debbie Stott

Why Increasing the Number of Compounding Periods Won’t Make You as Rich as You Might Think 26
Duncan Samson & Craig Pournara

An Alternative Trig Formula for Solving Triangles 31
Letuku Moses Makobe

Reflecting on a 2nd Round 2013 SA Mathematics Olympiad Problem 34
Michael de Villiers

A Quick Tool for Tracking Procedural Fluency Progress in Grade 2, 3 and 4 Learners 36
Debbie Stott

Timelines for Annuities – Getting to Grips with the Conventions 40
Craig Pournara

Why Still Factorize Algebraic Expressions by Hand? 44
Michael de Villiers

Nurturing Curiosity and Creativity through Mathematical Exploration 47
Duncan Samson

Erratum 54
From the Editors

Dear LTM readers

LTM14 begins with an interview with Rhodes University’s Deputy Vice Chancellor, Dr Sizwe Mabizela. Dr Mabizela is a renowned South African mathematician who enjoys “fiddling” or “doodling” with mathematical ideas. The interview engages him in a discussion around his passion for mathematics and mathematics education. In the second article in this issue, Mark Rushby presents and discusses a number of warm-up activities that he has found work well at a Grade 7 level and which have the potential to add to learners’ enjoyment of mathematics. The third article explores conceptual and visual aspects of the process of “completing the square”, while in the fourth article Mellony Graven and Hamsa Venkatakrishnan reflect on the Annual National Assessments (ANAs). Michael de Villiers then explores the idea that equality doesn’t always represent the best solution to a modeling problem, while the sixth article reviews a useful website that allows one to tailor-make number line images.

The seventh article explores why increasing the number of compounding periods won’t make you as rich as you might expect, after which Letuku Moses Makobe explores an alternative trig formula for solving triangles. The ninth article reflects on a question from the second round of the recent South African Mathematics Olympiad. Debbie Stott then shares a useful tool for tracking procedural fluency progress in Grade 2–4 learners. In the eleventh article Craig Pournara explains why conventions are so important when working with timelines in financial mathematics, after which Michael de Villiers presents a scenario which highlights the value of manual algebraic manipulation in a world where powerful computer algebra systems might seemingly be an easier alternative. The final article in this issue looks at investigations and mathematical exploration as a means of nurturing curiosity and creativity, essential components of a healthy mathematical disposition.

We hope you enjoy the wonderfully diverse array of articles in this issue, and remind you that we are always eager to receive submissions. Suggestions to authors, as well as a breakdown of the different types of article you could consider, are printed on the inside of the back cover of this journal. If you have an idea but aren’t sure how to structure it into an article, you’re welcome to email one of the editors directly – we’d be happy to engage with you about turning your idea into a printed article.

Duncan Samson, Marcus Bizony & Lindiwe Tshabalala
In November 2012 we interviewed Dr Sizwe Mabizela, our Deputy Vice Chancellor at Rhodes University, to find out about his love for mathematics and his enjoyment of “playing” or “doodling” with mathematics and mathematical ideas. Dr Mabizela is a renowned South African mathematician who has a deep concern about the teaching and learning of Mathematics. We were writing a paper (Graven & Schäfer, in press) for a book on mathematics knowledge for teaching entitled “Exploring Content Knowledge for Teaching Science and Mathematics” and wanted to make the point that often instilling a playful love of mathematics in learners is absent from the literature. We argue in the paper that a passion for mathematics and embracing a desire to explore and play with mathematics is as essential and important an ingredient for being an effective Mathematics teacher as content knowledge of the subject. We interviewed several of our favourite mathematics educators and asked them about how their love of mathematics played out in their life and how it all began. We share with you an edited excerpt from our extended interview with Dr Sizwe Mabizela and hope that it serves to inspire.

**SIZWE:** Doodling with mathematics is one of those things that I do all the time. It’s something that just comes naturally to me and which I really enjoy doing. It’s something that I find very engaging – it’s a pastime for me really.

**MARC:** What do you do when you play with mathematics?

**SIZWE:** Usually I’m creating interesting problems in mathematics, or I’ll be working on a research question. I’ve previously done research in functional analysis and there were some very interesting questions which came up from time to time, so when I have the time and space I sometimes re-visit those questions. And since I have the questions in my head wherever I go I often try to play around and see if I can make progress on those problems that I hadn’t managed to completely solve. So it’s just something that I really enjoy doing. It’s a very relaxing thing.

**MARC:** In what way does it relax you?

**SIZWE:** I don’t know, it just takes me to a different space. I just enjoy being in that space. I serve on the South African Mathematics Olympiad question committee, and those weekends we get together to formulate problems for the Olympiad are the best times for me – it’s very, very exciting, something that I thoroughly enjoy. Mathematics is something that’s truly amazing because sometimes you try to solve a problem and you don’t really make any headway on it so you just put it aside. And then there’s a period of incubation when nothing seems to be happening at all, but then you are astounded when you return to the problem and suddenly realise that the solution is so simple and straightforward. And when you look at it...
afterwards you say ‘of course it had to be!’ I’m often amazed by young people, particularly in the Mathematics Olympiad. I would sometimes take quite a while trying to solve a problem, really writing a very detailed solution to the problem. Then you’d get a young kid in Grade 9 who’d come up with a really clever, very innovative solution. And you say ‘well, that’s the beauty of mathematics!’

**MELLONY:** Where did your love of mathematics come from? Where did it all start?

**SIZWE:** That’s always a difficult question to answer because for as long as I can remember I’ve always enjoyed mathematics. It came very naturally. My mother always encouraged us to do mathematics, so from that early stage I really enjoyed it. When I was at boarding school I had an exceptional teacher who was a huge inspiration to many of us, and then at university I was taught by the remarkable Terry Marsh. After graduating, and throughout my teaching career at university, I so much wanted to be like him. He was a quintessential gentleman. I can still distinctly remember from my first year at university how he wrote those integrals and derivatives, the way they were laid out in a very systematic way. He had a huge influence on me in terms of my enjoying mathematics. So I always make the point that one can never over-emphasize the importance of having a good teacher, a caring teacher. So going back to the question you asked, I’ve always enjoyed mathematics.

**MELLONY:** In your teaching career, when you were teaching mathematics, how do you think this passion for mathematics manifested itself?

**SIZWE:** One thing that I tried to do was to connect with the students and to make them realise that mathematics is something that they could do. I’m not one of those people who believe that there are certain people who are cut out for mathematics and that there are those who aren’t. I know that there are issues of potential, aptitude and all that, but I still believe that everyone can develop a love and appreciation for mathematics. So I try to connect with the students. I think students are very perceptive – they can sense when you are genuinely enthusiastic and when you enjoy what you are doing. I can say that I really started knowing mathematics when I had to teach it. Although it all came relatively effortlessly to me when I was a student, when I had to explain it to my students it was a totally different ball-game. One of the beautiful things about mathematics is that it works on definitions, and students need to clearly understand these definitions. And that’s really what the beauty of mathematics is, because you can’t simply call on an authority and say ‘so-and-so said it, therefore it must be’. You need to be able to argue logically and coherently, using all the key things in mathematics to make your case. There is no authority except a valid mathematical proof, and I think that for me this is very important. So one of the things I’ve learnt from mathematics is this logical, systematic and critical way of thinking. In anything that I do I always try to separate the issues in order to see what is important and what is not important in terms of understanding the problem and working towards a solution.

**MARC:** Mathematical playfulness is clearly important in your life, perhaps as much as formal mathematical engagement. How do you marry these two aspects of mathematics, and how would you encourage or inspire children to play more with mathematics?

**SIZWE:** Informal interaction with mathematics and just being comfortable exploring mathematical ideas in a casual way is important. However, I don’t really make any distinction between informal and formal mathematics. For me mathematics is mathematics, wherever it is, it’s just a continuum. It’s part of me.

**MARC:** You’ve mentioned that mathematics generally came very easily and naturally to you. But what about those children or individuals who struggle with maths? How would you encourage them to play with maths or to engage with maths in a playful manner?

**SIZWE:** You have to start with where they are at, and this is where issues of ontological assumptions come into play. The first thing you have to understand is where they are in terms of their own development and appreciation of mathematics, and gradually nurture them through encouragement and reinforcement. It’s important to affirm what learners know because we acquire new knowledge by relating new things to what already exists in our knowledge schema. So, in the first instance one should affirm what learners already
know, and build from there. My own sense is that all too often we discount what learners already know. I strongly believe that every young person has the potential to develop their mathematical ability if properly nurtured and supported. There are some really brilliant kids out there. I can remember a particular incident when I was at UCT giving a talk on the beauty of mathematics to a group of learners as part of a weekend workshop. My presentation was on modular mathematics and how it plays a role in security and the security industry. I was explaining some of the basics of number theory that were needed for the security industry when this young Grade 7 boy put up his hand to ask a question. And what he asked I had never thought of. I was stunned. After some thought I was able to answer his question, but momentarily I was struck wondering ‘Wow, where did that come from?!’ I really enjoy moments like that.

**Mellony:** There are loads of wonderful mathematical games that you can download from the internet. Do you ever spend any time playing these sorts of games?

**Sizwe:** Oh yes, I enjoy them. I often get hooked, particularly Sudoku – the fiendish ones can be very challenging, so I enjoy doing that. There are also other games that require logical thinking, like the one where you have to move trucks around. Traffic Jams?

**Mellony:** Oh yes. The trucks are jammed and you have to move them around to free them. What amazes me is my five year old daughter is often better than me! She can often see the solution just like that!

**Sizwe:** I’m not surprised. I know someone has made this point before, but one of the major problems with our formal education is that it often kills creativity.

**Mellony:** Absolutely. Sir Ken Robinson famously stated just that on one of his TED talks.

**Sizwe:** Yes. I think there’s a great deal of truth in it. One gets into a formal way of looking at things and one’s creativity gets stunted. It becomes difficult to think outside the box. I think this is why young kids are able to see things in a different way – they are not constrained by the formalistic ways of doing things that we all become caught up in.

**Closing remarks**

Sizwe’s story allows us to see how mathematical playfulness is a deeply satisfying activity that is both recreational, enjoyable and closely connected with problem solving. He often thinks about how to enable our learners to discover the same sense of awe and wonder of mathematical discoveries or of struggling through problems. His interview suggests a dialectical relationship between a passion and love of mathematical exploration and a passion and love for nurturing and sparking this in others. These dual passions appear to seamlessly nurture each other. Of significance to mathematics teachers is that he identifies specific individual teachers who have played a significant role in shaping his teaching style as well as his attitude towards his students. He encourages teachers to believe in their students and to nurture them through encouragement and reinforcement – and by appreciating Mathematics.

We hope this story inspires mathematics teachers to play with maths and instill a love of mathematical play in their learners.

**Acknowledgments**

The work of the SA Numeracy Chair and the FRF Mathematics Education Chair, Rhodes University is supported by the FirstRand Foundation (with RMB), Anglo American Chairman’s Fund, the Department of Science and Technology and the National Research Foundation.

**References**


---

1 See Sir Ken Robinson ‘Schools kill creativity’ on: [www.ted.com/talks/ken_robinson_says_schools_kill_creativity.htm](http://www.ted.com/talks/ken_robinson_says_schools_kill_creativity.htm)
Warm-up Activities for Grade 7s

Mark Rushby
Sweet Valley Primary School, Bergvliet, Cape Town
deputy@sweetvalleyprimary.co.za

As teachers, we are usually keen to get into the day’s Mathematics lesson for a particular class right from the outset, often forgetting that pupils have typically just been engaged with a different subject and as such may need a little time to make the mental transition to the Mathematics classroom. A warm-up activity at the start of a Mathematics lesson is often a good way to help pupils make this transition. What follows are a number of activities that I have found work well with Grade 7 pupils.

**FOUR 4S**

Once the Grade 7s have been exposed to square roots and exponents, the well known “four 4s” challenge makes an excellent warm-up activity. The “four 4s” problem requires pupils to find mathematical expressions for as many whole numbers as possible using four 4s each time along with any of the symbols $+, - , \times , \div , \sqrt[n]{\cdot}$, ! and brackets. Fours may be ‘joined’, for example to make 44, and the symbols may be used as many times as you like. Although most Grade 7s will be unfamiliar with factorial notation, it is a simple matter to explain how the factorial symbol operates $[n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1]$. While some variations of the “four 4s” problem allow the use of the decimal point, I generally ignore this option.

Examples of expressions for the first four whole numbers are shown below, although there are certainly many other possibilities:

$$1 = \frac{4 \times 4}{4} \quad 2 = \frac{4 + 4}{4} \quad 3 = \frac{4 + 4 + 4}{4} \quad 4 = \frac{4\sqrt{4} + \sqrt{4} + \sqrt{4}}{4}$$

One way to use the “four 4s” problem as a warm-up activity is to challenge pupils to find as many different expressions as possible for a given whole number. By way of example, 2 can be expressed in the following different ways:

$$\frac{4 + 4}{4}, \quad \frac{4\sqrt{4} + 4}{4}, \quad 4 \times (4 - 4) + \sqrt{4}, \quad \frac{4!}{4} - \sqrt{4 \times 4}$$

I have used the “four 4s” problem every year for many years, and pupils have found the activity both accessible and enjoyable. This is a great warm-up activity for mixed ability classes.

**MAKE 24**

Having spent some time on the “four 4s” problem it is an easy transition to “make 24” using four given numbers, with the same rules as for the “four 4s” problem. It is easy enough to find examples of “make 24” cards on the internet, such as that shown alongside. Some pupils get very good at this activity and come up with some quite spectacular solutions. The following solutions for the card alongside were all generated by Grade 7 pupils in a warm-up activity:

$$2\sqrt{4} - 6 - \sqrt{4} \quad \frac{6!}{5 + (4 + 2)} \quad \frac{[5 - (6 - 2) + 4]}{4}$$

$$\frac{5!}{(6 + 4) \times 2} \quad \frac{6!}{5! \times 2 \times \sqrt{4}} \quad \sqrt{6!} - 5! - (\sqrt{4 + 2})$$
After about 10 minutes into the problem I invite pupils to write their solutions on the board in order to generate a class discussion around the various proposed solutions.

Rather than using prepared cards, one can simply present pupils with four random digits and challenge them to come up with ways of arriving at an answer of 24. While we obviously haven’t exhausted all the possibilities, no combination of digits has yet stumped any of my classes. I thought that finding expressions for the odd numbers from 1 to 29 using the four digits 2, 4, 6 and 8 might prove difficult, but the pupils rose to the challenge without any problem.

In their final examination of 2012 I put in the following problem:

<table>
<thead>
<tr>
<th>Using the digits 2, 4, 7 and 9, and any of the operations $+, -, \times, \div, \sqrt{\phantom{0}}, !$ and brackets, find five different expressions each equal to 1. In each case, each of the digits may be used only once. The digits may be joined to form larger numbers, e.g. 47.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$24 + \sqrt{9} - 7$</td>
</tr>
<tr>
<td>$\frac{9 + 4 + 2}{7}$</td>
</tr>
<tr>
<td>$\frac{2^4 - 9}{7}$</td>
</tr>
<tr>
<td>$2 - \sqrt{7} + 9$</td>
</tr>
<tr>
<td>$\frac{\sqrt{5} + 7}{2} - 4$</td>
</tr>
<tr>
<td>$2 - \frac{7}{4 + \sqrt{9}}$</td>
</tr>
<tr>
<td>$\frac{4! - \sqrt{9}}{7} - 2$</td>
</tr>
</tbody>
</table>

**Concluding Comments**

At the start of the year, many pupils are likely to write something like $7 + 4 = 11, 9 + 2 = 11, 11 \div 11 = 1$ rather than $1 = (7 + 4) \div (9 + 2)$. The activities discussed in this article have the added benefit of allowing the order of operations and the use of brackets to be continuously reinforced, thus encouraging pupils to write expressions succinctly and correctly.

It has been my experience that pupils respond really positively to the types of open-ended problems described here, and in addition to serving as good warm-up activities I have found they contribute substantially to pupils’ enjoyment of the subject.
“Completing the Square” – A Conceptual Approach

Duncan Samson
FRF Mathematics Education Chair, Rhodes University, Grahamstown
d.samson@ru.ac.za

The method of completing the square is a useful algebraic technique. Within the South African school context, “completing the square” is most often used for three specific purposes: (i) solving quadratic equations, (ii) writing parabola equations in turning point format, and (iii) writing circle equations in centre-radius format. In my experience, learners who have good algebraic skills master the technique of completing the square fairly quickly. Learners who are less algebraically confident take a little longer to acquire the skill but are nonetheless able to master the technique with sufficient practice. However, for most learners the method of completing the square is little more than an arcane process of algebraic manipulation accomplished somewhat mechanistically through the use of a guiding algorithm or mantra such as halve it, square it, add it to both sides. As such, most learners leave school without really understanding the conceptual basis of the technique. This is sad because the conceptual heart of the process is not only simple but beautifully elegant. A more conceptual approach to introducing learners to the technique of completing the square is likely not only to demystify the method for many learners, but is also likely to free learners from their reliance on rote algebraic manipulation. This article explores strategies for teaching the method of completing the square using a more conceptual approach.

THE BASIC CONCEPT

There is nothing fundamentally difficult about the technique of completing the square. The algebra can sometimes make the process rather messy, but I would argue that even a Grade 7 learner could use the basic concept, without recourse to any algebra, to solve a quadratic equation. Consider the following question:

If a rectangle has an area of 55 square units, and one side of the rectangle is 6 units longer than the other, how long is the shorter side?

Let’s begin by visualising the scenario:

\[
\begin{array}{c}
? \\
\hline
? + 6 \\
\end{array}
\]

\[\text{Area} = 55\]

\textbf{Figure 1:} Visualising the problem.

In order to determine the length of the shorter side, let’s reconfigure the rectangle as a square. Think of the rectangle as a square that’s been extended by 6 units in the horizontal direction. Take this “extension” and split it vertically into two rectangles. Leave one of these two rectangular pieces where it is, and move the other as indicated in Figure 2.
Once we’ve reconfigured the rectangle we can “complete the square” by adding in an additional piece to the bottom right-hand corner of the reconfigured rectangle (Figure 3).

The shaded area in Figure 3 is the reconfigured rectangle which the original question tells us has an area of 55 square units. The additional piece that completes the square is clearly a 3 by 3 square with an area of 9 square units. The completed square thus has an area of $55 + 9 = 64$ square units from which it follows that it has sides of 8 units. The length of the shorter side of the original rectangle is thus 5 units.

What’s important to understand about this process is its generality, and why the specific reconfiguration used will always result in a larger square once a smaller square has been added to the bottom right-hand corner. This of course is the conceptual heart of the process of “completing the square”, and therein lies the elegance of the technique. Visually it should become quickly apparent why this process will always work.
INTRODUCING ALGEBRA TO THE PROCESS

In the previous question we have in effect solved the quadratic equation \( x(x + 6) = 55 \), although we did so using a visual approach without recourse to any algebra. Although the quadratic equation \( x(x + 6) = 55 \) has two solutions, \( x = 5 \) and \( x = -11 \), since the original context of the question related to the dimensions of a rectangular shape we are only interested in positive answers. Let us now extend the basic concept but introduce a bit of algebraic symbolism.

Consider the following question:

If a rectangle has an area of 60 square units, and one side of the rectangle is 4 units longer than the other, how long is the shorter side?

Let's begin by once again visualising the scenario:

After reconfiguring the rectangle and completing the square we have now generated a new square with sides of length \( x + 2 \). Since the shaded area (i.e. the original rectangle) has an area of 60 square units, and the completion of the square has increased the area by 4 square units, the area of the completed square is thus 64 square units, or expressed algebraically, \( (x + 2)^2 = 64 \). From this it follows that the sides of the square must be 8 units long (i.e. \( x + 2 = 8 \)), from which it follows that the length of the shorter side of the original rectangle must be 6 (i.e. \( x = 6 \)). Once again we are only interested in positive solutions based on the context of the original question.
What’s important to understand now is how the size of the additional square in the bottom right-hand corner relates to the algebra that has been introduced to model the scenario. Visually it should be readily apparent why the sides of the small square will always be half the co-efficient of the $x$ term in the original expanded equation, i.e. $x^2 + 4x = 60$ in this particular case. From this it should be equally apparent why the area of the small square will always be the square of half the co-efficient of the $x$ term in the original expanded algebraic equation.

**Moving away from the visual model**

Thus far we have solved a quadratic equation using a visual approach without any recourse to algebra. We then expanded on this idea by introducing algebraic symbolism and establishing important conceptual connections between the visual and algebraic analogues of “completing the square”. Up until now we have only been interested in positive solutions, so we now need to make a final step of moving away from the visual model in order to consider both positive and negative solutions to any given quadratic equation.

Let’s consider the following decontextualised algebraic equation: $x^2 + 7x = 58$. Based on our conceptual understanding of the visual process of completing the square, we can immediately see in our mind’s eye the “completed square” with sides of length $x + 3\frac{1}{2}$ as well as the additional square in the bottom right-hand corner with area $(3\frac{1}{2})^2$. This immediately gives us the equation:

$$(x + 3\frac{1}{2})^2 = 58 + (3\frac{1}{2})^2$$

$$(x + 3\frac{1}{2})^2 = 70\frac{1}{4}$$

Since the original question was a decontextualised algebraic equation, we can now use our understanding of squares and square roots to continue the algebraic solution to find values of $x$ that make the equation $x^2 + 7x = 58$ true:

$$x + 3\frac{1}{2} = \pm \sqrt{70\frac{1}{4}}$$

$$x + 3\frac{1}{2} = \pm 8.38...$$

$$\therefore \quad x = 4.88 \text{ or } x = -11.88 \quad \text{(to two decimal places)}$$

**Concluding comments**

In this article I have focused on teaching strategies that promote the development of a conceptual understanding of the method of completing the square. Once a firm conceptual understanding has been established, learners should have a much deeper appreciation for what many no doubt see as an abstract and mechanised piece of algebraic manipulation. Introducing further levels of complexity, for example dealing with scenarios where the leading co-efficient isn’t 1 or where the co-efficients are letters rather than numbers, should also be far easier to navigate once a conceptual basis has been established. This in turn should lead to a greater appreciation of the quadratic formula, and demystify what for many learners is simply a rote process of substitution. Using the fundamental concept as a basis for writing parabola equations in turning point format and writing circle equations in centre-radius format should also be very helpful. But perhaps most importantly, a conceptual understanding of the technique of completing the square makes critical links to the historical development of the process and exposes learners to both the elegance and beauty of a wonderful piece of mathematical ingenuity.

**Acknowledgement**

The work of the FRF Mathematics Education Chair, Rhodes University is supported by the FirstRand Foundation Mathematics Education Chairs Initiative of the FirstRand Foundation, Rand Merchant Bank and the Department of Science and Technology.
ANAs: Possibilities and Constraints for Mathematical Learning

Mellony Graven\textsuperscript{1} & Hamsa Venkatakrishnan\textsuperscript{2}
\textsuperscript{1}SA Numeracy Chair, Rhodes University \textsuperscript{2}SA Numeracy Chair, Wits University
\textsuperscript{1}\texttt{m.graven@ru.ac.za} \textsuperscript{2}\texttt{hamsa.venkatakrishnan@wits.ac.za}

The introduction of the Annual National Assessments (ANAs) began in 2011. The ANA was explicitly focused on providing system-wide information on learner performance for both formative purposes, such as providing class teachers with information on what learners were able to do, as well as summative purposes, such as providing progress information to parents and allowing for comparisons between schools, districts and provinces (Department of Basic Education (DBE), 2011). The ANAs were written by all government school learners in Grades 1-6 as well as Grade 9 in September 2012. The ANAs focused on Literacy and Numeracy in the Foundation Phase, and Language and Mathematics in the Intermediate Phase. The 2012 national report of the ANAs (DBE, 2012) is available for downloading at http://www.education.gov.za.

Assessments such as the ANAs of course have an influence on what happens in schools and in classrooms. In our work as the SA Numeracy Chairs at Rhodes and Wits University respectively we collaborate with teachers in 22 primary schools, of which 12 are located in the broader Grahamstown area and 10 in the Johannesburg area. The schools represent a mixture of both township and suburban schools in both Chair projects. Across both our projects we found during 2012 that several weeks of school time were taken up with the preparation and writing of the ANAs. A range of 1 to 8 weeks (with a mean of 3.97 weeks) were reported by our teachers to be taken up on the ANAs. In our respective teacher development programmes (namely, the Numeracy Inquiry Community of Leader Educators (NICLE) and the Wits Maths Connect - Primary) our teachers shared a range of different experiences of the ANAs. Together with the teachers we decided that it was important to capture/document the range of views and experiences by gathering this data in the form of questionnaires that teachers completed across our two projects. The questions asked related to the following range of issues concerning teacher experiences of the ANAs:

- the purpose and value of the ANAs
- the use and value (if any) of exemplar papers given before the ANAs
- the administration of the ANAs
- the marking of the ANAs
- teaching time taken up by ANAs (including preparation, administration, marking and preparing results)
- correspondence with topics taught by teachers
- the extent to which ANAs reflect learners’ mathematical/numeracy competence
- any other experiences/issues in relation to the ANAs

Participation in the questionnaire was voluntary. 54 teachers from across 21 schools completed the questionnaires. Here we share with you the various key themes that emerged, including examples of what teachers wrote in relation to these themes. Thus, rather than discussing the entire range of data received, we primarily share those recurrent experiences that teachers communicated. We believe that dialogue is important in relation to the effect of the ANAs and we hope that this paper will stimulate teachers in other schools and districts to get together to share their experiences and then to feed these experiences back to districts. Our hope is that this will support the ongoing reflection and revision of the ANA process.
A note at the outset is that these views are by no means considered representative of the general population of teachers. Rather, we hope that the data we share will serve as a stimulus for further engagement and discussion among teachers as to the extent to which these experiences resonate with or depart from their own experiences.

In the table below we provide an overview of comparative positive and negative recurrent responses across a range of issues relating to teacher experiences of the 2012 ANAs:

<table>
<thead>
<tr>
<th>Positive points</th>
<th>Negative points</th>
</tr>
</thead>
<tbody>
<tr>
<td>ANAs are good for:</td>
<td>Language within questions blocks access to question meaning for learners with weak reading and writing skills (this was a particular issue at Grade 3 level). This in turn is linked to lack of time for paper completion for weaker learners.</td>
</tr>
<tr>
<td>- Standardizing content coverage</td>
<td>Learners needed some ‘explanation’ of the task in order to access the question, thereby disrupting the validity of assessment of learner understanding.</td>
</tr>
<tr>
<td>- Making explicit one’s expectations about what will be assessed</td>
<td>Predominant view of strong correspondence between content coverage in class and ANA questions.</td>
</tr>
<tr>
<td>- Providing information on learners’ levels of understanding</td>
<td>ANA timing in September results in difficult and rushed 4th term content coverage or alternatively in non-alignment with content coverage.</td>
</tr>
<tr>
<td>- Providing guidance on content coverage</td>
<td>Only one positive comment related to the smooth administration of the ANAs in a particular teacher’s school.</td>
</tr>
<tr>
<td>Predominant view of strong correspondence between content coverage in class and ANA questions.</td>
<td>Bureaucratic arrangements (monitoring another class, seating arrangements within classes, lack of reading out of and explaining questions) seen as anxiety inducing for Foundation Phase learners in particular. Disrupts duty of care.</td>
</tr>
</tbody>
</table>

Examples of the kinds of positive points summarized above are as follows:

‘The values and purpose of ANA are good because they help educators to do curriculum pacing very well and to cover the content prescribed for that class or grade’ (Gauteng teacher)

‘Good. They will standardize the content for each grade.’ (EC teacher)

‘ANA is a good tool to test our learners’ ability on how well they are doing in mathematics’ (Gauteng teacher)

‘The purpose is to assess the learners and to ensure that content of work is covered. And to see where problem areas are.’ (EC teacher)

The following comments, whilst buying into the purpose of the ANA, raise implementation issues:

‘[The purpose of ANA is] to see if learners know the work and understand it. We need what you call pacesetters at the beginning of the year. The paper is based on the whole year’s work. Some of the work we did not cover yet because we are left with the fourth term still.’ (EC teacher)

‘I think the ANA will be more successful if they had given the pacesetters in the beginning of the year plus example questions. I did not like the idea that we had to facilitate other grades instead of staying in my class.’ (EC teacher)
Noting similar concerns in more negative ways, one teacher responded as follows:

‘They were not useful because they cover the whole year’s work in September; I can't rush to finish everything in September, because in that way I will be teaching the syllabus, not the learners.’ (EC teacher)

Language issues were also raised, particularly in relation to learner difficulties with reading and writing demands, and the consequences of this for anxiety:

‘ANA is confusing learners, because Grade 1 to Grade 3 [learners] are very small they are used to their teachers explaining for them so ANA does not allow the teachers to read the instructions for the learners, especially Grade 3. These learners are small - they still need guidance when writing exams.’ (EC teacher)

Of interest, several Eastern Cape teachers pointed to the differential value of the ANA for weak and strong learners. For weak learners, comments related to weak reading and writing skills. For example:

‘The 'clever' kids did it with ease, but some learners whose writing and reading is poor needed help.’ (EC teacher)

‘It helped the clever kids, but for those with writing and reading problems it was not easy as they took long to read and write.’ (EC teacher)

Perhaps some of these comments relate to some of the raw data provided by some NICLE teachers which show that several learners achieved 0% for the ANAs across several grades, indicating an inability to access what was required of them. This was not however the case on alternative orally administered numeracy tests that were administered within the broader research project.

Related to the above were a wide range of comments and phrases relating to how the administration of ANAs led to learner anxiety and teacher frustration at not being able to provide care for their learners, particularly in the Foundation Phase, as teachers were not allowed to be present in administration of ANAs to their own classes. Phrases such as: ‘learners were very anxious/agitated/nervous and scared’ came up repeatedly. Teachers expressed frustration, using phrases such as: ‘my mind was thinking about my own class as there was a stranger in front of them’, ‘some learners become nervous with a new teacher in their class’.

We now turn our focus to the ANA exemplar papers. The table below summarizes the positive and negative responses in relation to the provision of exemplar ANA papers and the marking memorandums.

<table>
<thead>
<tr>
<th>Exemplar papers</th>
<th>Positive</th>
<th>Negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Useful for:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>- revision of content</td>
<td></td>
<td></td>
</tr>
<tr>
<td>- getting learners familiar with the format of the ‘exam’ and the style of questions; helpful for dealing with learner anxiety</td>
<td></td>
<td></td>
</tr>
<tr>
<td>- preparation for ANA ‘exam’ as high degree of overlap between exemplars and ANA mentioned quite frequently</td>
<td></td>
<td></td>
</tr>
<tr>
<td>- providing teachers with guidance on content coverage</td>
<td></td>
<td></td>
</tr>
<tr>
<td>- some reports of improving performance</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Difficulties attributed to:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>- Need to rush through exemplars</td>
<td></td>
<td></td>
</tr>
<tr>
<td>- Reading exemplars problematic for many learners</td>
<td></td>
<td></td>
</tr>
<tr>
<td>- Bureaucratic difficulties with photocopying and access to paper</td>
<td></td>
<td></td>
</tr>
<tr>
<td>- Some reports of learners not improving in spite of exemplars</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Learning and Teaching Mathematics, No. 14, 2013, pp. 12-16
In particular we note that several teachers commented on the high degree of similarity between the exemplar ANA questions and the final paper questions (some as positive and others as negative comments). For example:

‘Learners benefitted a lot from the [exemplar] questions because some of them were in the final exams’ (Gauteng teacher)

‘Was useful to use because some of the questions repeated to 2012 ANA question paper.’ (Gauteng teacher)

‘Just a duplicate of the ANA papers’ (EC teacher)

‘Some, they were useful because they were asked in the pre-ANA & the ANA.’ (Gauteng teacher)

A further problem relates to singularity of methods that were viewed as ‘acceptable’ and provided as correct answers in the memos:

‘They were useful because they set a good example of the exact way in which questions were to be asked so it trained my learners.’ (Gauteng teacher)

‘Multiplication in Grade 3 was difficult to mark as [a specific] method was given on the memorandum and learners did use different methods taught.’ (Gauteng teacher)

Concluding remarks

The power of the influence of national assessments on the teaching of learners, not to mention the teaching time that is given to these assessments, should not be underestimated. Four key issues are raised that we have forwarded to departmental structures for consideration. These relate to:

- the reading of questions for Grade 3s
- the timing of the ANAs and the relationship to content coverage
- the importance of breadth of questions (not to be confused with content coverage)
- acceptance of a variety of correct methods

We elaborate briefly on these issues below.

The decision that Grade 3 Foundation Phase learners do not have the questions read to them (as is the case with Grade 1 and Grade 2 learners) was raised by teachers across both projects as problematic in three ways: (i) learners’ poor language proficiency, (ii) learners’ poor reading skills (i.e. access to what is required), and (iii) care for learners (see previous quotes). The issue of care was particularly pronounced for Grade 3 learners who were not used to assessments such as the ANAs. Other teachers coming in to
assess them and not having teachers mediate what learners were required to do through verbal instruction or reading of questions were noted as particularly problematic. The quote below captures this:

“Anxiety was a big factor. Children were nervous. Learners' behavior was different as when writing internal tests/exams. I did not like the fact that we did not facilitate our own classes. We needed to shift classes. Children were confused. Especially Foundation Phase learners. Foundation Phase learners need their own educators. I neglected my own assessment for third term… Learners had 'exam fear'! Poor learners!” (EC teacher)

On the second point, the timing of the ANAs must be chosen to correspond with what teachers can be expected to have covered by the time of writing. Additionally, some teachers indicated the wish to be given clear guidelines (or pace-setters) at the start of the year. A number of quotes indicate teacher frustration with not being able to complete all the work by September.

On the third point, relating to the breadth of question, given the widespread acceptance of the usefulness and purpose of the ANAs (as evidenced by previous quotes) special attention must be given to the influence of these assessments on classroom practice. Teacher utterances largely indicate acceptance of ANA questions as valid exemplars of ‘the’ appropriate standard, format, scope and coverage expected of teachers in relation to their teaching. Thus we argue that extremely careful consideration must be given to the choice of questions ensuring both range in format, style, scope and content if we are to avoid a situation of teaching becoming limited to what is assessable within a limited time ANA assessment. Thus ‘reverse recontextualisation’ (Barbosa, 2013), that is considering the imagined effect of the ANAs in the classroom, must be considered. Should ANA exemplars and ANAs over the years be too similar across style, scope, content and format each year there is a danger that while we will see improvements in performance these improvements will not necessarily be matched by improved mathematical learning and competence, and several key processes and skills (such as mental arithmetic and investigative problem solving) could disappear from classrooms.

Additionally, with regard to the fourth point, teachers raised concerns that the ANA exemplar and paper memos did not accept alternative methods for working with calculations. This is problematic given the research evidence that multiple representations are an important part of mathematical learning. In practical terms it is also highly discouraging for teachers and learners to be marked down for answers that have been correctly produced.

On a final note we hope and trust that this paper stimulates constructive deliberations in the ongoing review of the ANA process.

Acknowledgement:
The work of the SA Numeracy Chairs is supported by the FirstRand Foundation (with the RMB), Anglo American Chairman’s fund, the Department of Science and Technology and the National Research Foundation.

References:
Equality is Not Always ‘Best’!

Michael de Villiers

*University of KwaZulu-Natal*

*profmd@mweb.co.za*

Mathematical modeling as an important mathematical skill or process is emphasized in the 2011 Curriculum and Assessment Policy Statement (CAPS) for Mathematics in South Africa as follows:

“Mathematical modeling is an important focal point of the curriculum. Real life problems should be incorporated into all sections whenever appropriate. Examples used should be realistic and not contrived. Contextual problems should include issues relating to health, social, economic, cultural, scientific, political and environmental issues whenever possible.” (Department of Basic Education, 2011, p. 11)

**THE MODELING PROCESS**

As discussed in De Villiers (2007), the process of mathematical modeling essentially consists of three steps or stages as illustrated in Figure 1, namely (i) construction of the mathematical model, (ii) solution of the model, and (iii) interpretation and evaluation of the solution. Traditionally, much of schooling consisted of teaching learners how to solve given mathematical models such as linear, quadratic, trigonometric, exponential, and hyperbolic functions, hardly involving them at all in the construction and development of these models from real world contexts, and much less on evaluating the solutions and how well they fit ‘reality’. In short, it consisted of teaching mathematics largely disconnected from the real world.

In traditional geometry teaching at the GET (Grades 7-9) and FET (Grades 10-12) Mathematics levels in South Africa, this disconnection has been even worse, with teaching and textbooks ignoring applications almost completely, and to a large extent only developing the geometry content with the idea to engage learners with ‘rigorous’ proof in a deductive system. So the intent of the new CAPS curriculum to promote mathematical modeling is welcome, and at last bringing South Africa more in line with trends in other countries. However, despite the laudable statements about mathematical modeling in the new Mathematics CAPS quoted at the beginning of this article, one wonders to what extent a genuine effort will be made to connect geometry to the real world, especially since the CAPS documents do not provide any examples of possible real world contexts.

In all the university courses I teach to prospective mathematics teachers, I try to bring in mathematical modeling as much as possible and constantly expect students not only to develop their own mathematical models, but also critically to evaluate the sense of the solutions provided by their models.
**EVALUATION OF A SOLUTION**

In Makae et al. (2001, pp. 171-180) and De Villiers (2003, pp. 27-33) the concept of *perpendicular bisector* is introduced as ‘a locus of equidistant points from the endpoints of a line-segment’ in order to find suitable equidistant points from four and three rural villages\(^2\) respectively to build a water reservoir. Associated with solving these two problems is the simultaneous introduction of the concepts *circumcentre* (as a point, if it exists, equidistant from \(n > 2\) points) and *cyclic polygons* (as polygons whose perpendicular bisectors of the sides are concurrent at the circumcentre; and therefore their vertices lie on a circle). Instead of using congruency as in the traditional proof, the concept of equidistance is then also used to logically *explain* (prove) *why* the perpendicular bisectors are concurrent for a triangle.

During 2011 I was in class with my prospective mathematics teacher students reflecting on the investigation for four and three villages that the students had completed in the previous session in the computer lab. I had open on my computer, and projected on screen, a triangle (ABC) with circumcentre (O) constructed with *Sketchpad* as the required equidistant point for the reservoir. To revise a question on the worksheet I dragged the triangle into a right-angled triangle and then continued dragging until it became obtuse, so that the circumcentre (O) moved far outside the triangle formed by the 3 villages (A, B and C) as shown in Figure 2.

I then repeated the question from the worksheet: “What did you notice about the position of the circumcentre … the desired position for the reservoir … when the triangle is acute, right and obtuse?” to which students quickly responded that it would for each case respectively lie inside the triangle, at the midpoint of the hypotenuse, and outside the triangle. Then I remarked, gesturing to the projected diagram for the obtuse triangle, something to the effect of: “Isn’t it surprising that the equidistant point for building the reservoir in the obtuse case lies completely outside the triangle?” At that point, one of my students, Xolile, shook his head and responded: “Eish! But that would not be a good position to build the water reservoir …” And immediately I responded by shrugging and saying: “But that’s the circumcentre … the unique point exactly the same distance from all three villages, don’t you agree? … as was the requirement of the problem right at the beginning. That is fair, isn’t it: that the villagers from each village walk equal distances?” However, Xolile continued to shake his head and said: “Eish, but then they have to walk too far …” “Yes, Xolile”, I agreed. “They obviously now have to walk further than when the triangle is acute. But they are all still equal distances from the reservoir; that’s only fair, isn’t it?” However, not wanting to just dismiss him, I paused for a moment and asked: “Well, where do you feel the reservoir should instead be built when the triangle is obtuse?” To this, Xolile responded: “I think the reservoir should not be built outside the triangle because then they have to walk too far …”

Only then did it hit me what he meant, and that he was perfectly right! It was visually obvious in the diagram! Something neither I nor any student from previous classes had raised before or critically reflected

---

\(^2\) It was assumed at the outset that the villages were all of equal size, while students were expected to identify other assumptions such as that the villages were on the same plane, that the surface was flat, that there were no natural obstacles such as dense forest or deep dongas, and that the distances were sufficiently large in comparison to the sizes of the villages and the water reservoir that they could geometrically be represented by points.
upon. If distance was the over-riding factor, the circumcentre was perhaps not the ‘best’ choice in the obtuse case. Even though the circumcentre was the ‘fair’ solution in being equidistant from all three villages, it seemed rather silly to have to walk so far for water!

In the subsequent discussion with the students, the following ‘better real world’ solution was arrived at. Consider any point $X$ on the perpendicular bisector of $AC$ as shown in Figure 2. Clearly, the closer $X$ comes to the triangle, the shorter the equal distances ($AX$ and $CX$) from villages $A$ and $C$ to $X$ will become, and these distances will become a minimum when $X$ reaches the midpoint $D$ of $AC$, and would increase again if $X$ went past $D$. Since the midpoint $D$ results in the shortest possible equal distances between villages $A$ and $C$, midpoint $D$ would obviously be the preferred choice for the reservoir between these two villages$^3$. Moreover, village $B$ will be quite happy with that choice and not complain, because $D$ is much closer for them than building the reservoir at $O$. Of course, the villagers from village $B$ will be advantaged over the villagers from the other two in walking a shorter distance to the water$^4$. But if villages $A$ and $C$ complain and put their foot down and insist on absolute equality and fairness, then they will have no choice but to have the reservoir built at $O$. But this will mean that everyone will have to walk much further than the choice at $D$! So only if they are willing to cooperate and compromise on the absolute insistence of equality in terms of distance will they benefit and be able to choose $D$ - with the result that everyone would be walking a substantially shorter distance. So perhaps surprisingly in this case, equality is not the ‘best’ choice they could make, and agreeing instead to ‘violate’ equality would provide an improved ‘real world’ solution!

**SOME FURTHER EXTENSIONS AND EXPLORATIONS**

**Cyclic quadrilateral**

In the same way as above, we could argue that in the case of four villages of equal size forming a cyclic quadrilateral$^5$, the ‘best’ choice of the water reservoir when the circumcentre $O$ lies outside the cyclic quadrilateral (as in Figure 3) would lie at the midpoint $E$ of the longest side $BC$.

However, someone might persuasively argue that if one just considers obtuse $\triangle ABD$, then the best choice would be the midpoint of $BD$. Similarly, the best choice for obtuse $\triangle ADC$ is at the midpoint of $AC$. Wouldn’t a better choice for the reservoir perhaps be the midpoint $H$ of these two midpoints rather than $E$? After all, if one looks at the displayed measurements, three of the villages ($A$, $B$ and $D$) would have shorter walking distances than to $E$, and only village $C$ would have a longer walking distance than before. Doesn’t that seem more ‘fair’? However, if we examine and compare the distances more closely, it is immediately noticeable that the walking distance from village $C$ to $H$ is more than twice

---

$^3$ Formally, $DC < XC$ because $XC$ is the hypotenuse of right $\triangle XDC$ (which itself follows from the theorem that the longest side in a triangle is opposite the largest angle), with the minimum of $XC$ clearly occurring when $X$ coincides with $D$.

$^4$ Though visually ‘obvious’ from the given diagram, $BD < AD$, since angle $BAD < angle ABD$.

$^5$ Unlike triangles, not all quadrilaterals are cyclic; so if the 4 villages do not form a cyclic quadrilateral the ‘best’ choice of position can be found by minimizing the sum of the differences (pairwise) between all the distances from the reservoir to the 4 villages (see De Villiers, 2003, p. 151). This can be done by measurement and dragging in Sketchpad, or algebraically, by applying the least squares method.

*Learning and Teaching Mathematics, No. 14, 2013, pp. 17-21*
the walking distances from villages $A$ and $D$ (and the walking distance of village $B$ is also substantially more than those from villages $A$ and $D$). In contrast, when we look at the walking distances from the four villages to $E$ and compare them, we ‘see’ that they are much closer in value. In other words, there’s a bigger ‘difference’ or ‘discrepancy’ between the comparative walking distances from the villages to $H$ than between the comparative walking distances from the villages to $E$.

Mathematically, we can quantify the above by finding the sum of the absolute values\(^6\) of the differences between all the distances from the villages to respectively points $H$ and $E$ to see which one gives the minimum value. As shown by the calculations in Figure 3, it’s clear that this ‘total discrepancy’ between the walking distances is less for point $E$ than $H$. Hence, $E$ ought to be the preferred choice.

**Building pipelines**

The context of building a water reservoir for 3 or 4 (or more) villages provides a rich context for mathematical modeling and problem solving. In Makea et al. (2001, pp. 181-186) the original problem is extended to selecting representatives from the four villages onto a Water Board, providing an opportunity to discuss some elementary mathematical issues surrounding ‘apportionment’ in order to obtain ‘fair representation’ if the four villages are of different sizes (also compare Nielsen & De Villiers, 2012).

Suppose next a nature conservationist on the eventually elected Water Board suggests that perhaps it might be better for the environment to rather build pipelines from the water reservoir to the villages, as not only would that save the villagers the effort of walking, but also prevent the formation of footpaths that can lead to serious vegetation and soil erosion (compare Makea et al., 2001, pp. 187-189). From a cost perspective, the problem now translates into minimizing the sum of the distances to the water reservoir to the villages (see Figure 4).

![Diagram](image-url)

**Figure 4:** Minimizing the sum of the distances.

From several experiences with learners at both school and university (undergraduate and postgraduate) I’ve noted that when confronted with this problem they initially tend to believe that the ‘best’ solution would still be to try and find an equidistant point from the four villages (thereby presupposing the existence of a circumcentre that would of course only exist if the four villages were concyclic). It thus usually comes to them as a great surprise and learning experience when, investigating dynamically, they find that the solution for a (convex) quadrilateral lies at the intersection of the diagonals\(^7\) (which is then easily explained (proved) using the triangle inequality).

Instead of a water reservoir, Häkköniemi, Leppäaho and Francisco (2011, p. 31) used the context of building an amusement park together with an open-ended approach with Grade 9 learners as follows:

> “Four towns plan to build together a magnificent amusement park. Investigate using GeoGebra what would be the most optimal and fair location for the amusement park.”

In such an ‘open’ approach learners are free to suggest and propose the kind of solution they consider the ‘best’. Interestingly, some learners came up with the equi-distant (circumcentre) proposal, while others suggested minimizing the sum of the distances to the amusement park.

---

\(^6\) Instead of taking absolute values, the differences could be squared (as in the ‘least squares’ method in statistics). The reason for making all the differences positive is to avoid having negative differences and positive differences canceling each other out.

\(^7\) For a concave quadrilateral when the diagonals fall outside, the minimum sum of distances to the vertices is found at the vertex with the reflex angle (compare Makea et al., 2001, pp. 188-189).
**Going 3D**

Lastly, one could also explore the analogous case of finding a suitable equidistant point for say a space-station for 4 planets in space forming a tetrahedron. Such an activity would naturally lead to the discovery (and proof) that, just like the triangle, the tetrahedron has a circumcentre and an associated circumsphere (compare Camou et al., 2013, pp. 68-71). However, now a similar problem as with the triangle arises when the tetrahedron becomes obtuse and the circumcentre falls outside the tetrahedron (see Figure 5). Since area in 3D is the analogue of length in 2D, we might, analogously to the obtuse triangle, expect the ‘best’ solution for an obtuse tetrahedron to lie at the circumcentre of the largest triangle lying opposite the largest dihedral angle. But what happens if that triangle itself is also obtuse? Does the ‘best’ solution then lie at that triangle’s longest side (edge)? These questions are deliberately left open to the reader to investigate further. A dynamic tool like *Cabri 3D* might be useful to conduct such an exploration.

**CONCLUDING REMARKS**

The episode with Xolile and the obtuse triangle and circumcentre rather nicely demonstrates that not only is mathematics never a precise model of the real world due to various simplifying assumptions that inevitably have to be made, but also that other practical considerations may indicate various different options that might be ‘better’ than the ‘purely’ mathematical solution. In contrast to ‘pure’ mathematics and its ‘neat’ solutions, ‘applied’ mathematics tends to be more ‘untidy’ with the need to interpret solutions given by mathematical models in a practical light. The episode also provides a good pedagogical example of how our students may sometimes teach us something new, provided we give them the opportunity to voice their opinions, and really listen to and try to understand what they have to say, especially if it doesn’t agree with our own pre-conceived ideas.

**REFERENCES**


Number Line Image Generator – A Website Review

Debbie Stott
South African Numeracy Chair, Rhodes University, Grahamstown
d.stott@ru.ac.za

http://www.oliverboorman.biz/projects/tools/number_lines.php

INTRODUCTION

Last year, as part of the South African Numeracy Chair, I was involved in creating a supplement for the local Grahamstown newspaper (Grocott’s Mail) called Fun with Maths. The aim of the supplement was to encourage parents and teachers to engage learners with various numeracy concepts in a fun way that differed from traditional teaching approaches. Supplements 4 and 5 of the series presented number line activities. Sizeable evidence shows that the number line is a powerful learning tool for children in primary school (see for example Beishuizen, 1997; Bobis & Bobis, 2005; Clarke, Downton & Roche, 2011). Researchers believe that regular use of number lines can develop learners’ ability to form a mental number line, which in turn may assist learners in carrying out mental computation tasks. However, research has found that many learners are unsuccessful in using number lines effectively, a fact which may well be attributed to their lack of experience engaging with number lines. This review provides a brief background of various types of number lines, particularly structured number lines, and describes and reviews a web-based resource that could be used to produce a variety of closed number lines for classroom situations. The number lines incorporated in the Fun with Maths supplements were all created using this web-based tool.

TYPES OF NUMBER LINES

‘Traditional’ or more familiar number lines with marked line segments (tick marks) are referred to as structured number lines where the numbers are representations of lengths or proportion, the lengths between the numbers and marks providing a model for learners to work from. Examples of structured number lines are given below.

![Examples of structured number lines](http://www.oliverboorman.biz/projects/tools/number_lines.php)

The empty number line (ENL) on the other hand is presented with no numbers or markers and is a “visual representation for recording and sharing students’ thinking strategies during mental computation” (Bobis, 2007, p. 410). The ENL, in which learners only mark those numbers they need for their calculation, allows learners to model their own thinking and to develop flexible mental strategies for solving problems (Bobis & Bobis, 2005).

---

8 These are available from Grocott’s series 1–10 at http://www.ru.ac.za/sanc/numeracyresources/grocottsupplement/#d.en.71044
Because they are used for different purposes, structured number lines and empty number lines are profoundly different. In order to use them effectively, learners will need to have different skills and knowledge (Bobis, 2007). Although some foundational skills and knowledge can be derived from working with the structured number line that can assist learners in understanding and using the empty number line, we cannot assume that learners who understand how to use one type of number line will automatically transfer that knowledge to working with the other (Bobis, 2007).

**Benefits**

Researchers believe that learners can benefit from working with number lines, not only using them to locate numbers, but using empty number lines to solve addition, subtraction and multiplication computations, as well as using number lines in problem-solving activities. The number line has the advantage of showing how numbers continue in both directions as far as you like, as well as providing a way to connect whole numbers, fractions and decimals (Gravemeijer, 1994). Additionally, Dehaene (2011) argues that our mental models of number seem to be linked to the ordinal aspects of number, where numbers are placed in order with respect to each other as represented on a linear scale or number line.

While structured number lines may not be useful for the flexible modelling of one’s own thinking, they do require that learners think in different mathematical ways about the example they are working with. For example, in Figure 1b) learners may start thinking in fractional distances between 0 and 1, while in Figure 1c) mathematical thinking (proportional reasoning) may be in terms of 10s, 20s, halving strategies to find the mid-point, and so on.


There are a variety of websites that generate number line worksheets for particular purposes - addition, subtraction, multiplication, or number representation (fractions, decimals, time and so on). Here I would like to explore and review one particular website that allows you to create your own fully customised number lines. Once created, you can download an image of the number line and insert it into documents to present as worksheets or activities for your learners. To give you an idea of the types of number lines you could produce using this website, I have included some examples below.

![Number lines generated on Number Line Image Generator.](http://www.oliverboorman.biz/projects/tools/number_lines.php)

**Figure 2:** Number lines generated on Number Line Image Generator.
### Using the Generator

First navigate to the website address (http://www.oliverboorman.biz/projects/tools/number_lines.php). The main part of the default landing page is shown in Figure 4.

The website page is very user-friendly, and a quick bit of playing around is probably all that’s required to get the hang of the various input options. The main input options are described below:

- **Start Value**: This setting determines where your number line will start. You can input both negative and positive values, including decimals.

- **Number of Ticks** and **Tick Difference**: Rather than entering an end value for your number line, the end point is automatically determined by your choice of “Number of Ticks” and “Tick Difference”. So, for example, choosing 5 ticks with a tick difference of 3 will result in an end value 15 units to the right of the start value. Only these tick values, along with the start and end values, will be labeled with numerical values.
• **Number of Subticks:** This setting determines the number of intervals into which the region between consecutive ticks will be divided.

• **Arrow Value:** This setting allows you to choose if you wish to have an arrow pointing to a specific point on the number line. To get rid of the arrow, either choose a value outside the range of your number line, or alternatively choose the “transparent” option from the “Arrow Colour” setting.

• **Prefix:** This setting allows you to customise the tick labels by specifying a prefix such as R or $.

• **Suffix:** This setting allows you to customise the tick labels by specifying a suffix such as %.

• **Show Continuation Arrows:** This option switches on/off the display of continuation arrows at either end of the number line.

The default landing page for the Number Line Image Generator has some of these input options pre-populated, as shown in Figure 4. Simply change these values to create your own customised number line.

Once you have created your number line you can then download it as an image in PNG, JPG or GIF format. It doesn’t really matter which you choose if you are working with print materials, so I normally choose JPG. Select the “Download as JPG” option just below the number line image and a dialogue box will open up giving you the option of either opening the image immediately or saving it to a folder. Once the image has been downloaded you are free to use it as you would any other image or picture.

**CONCLUDING COMMENTS**

The Number Line Image Generator website is relatively easy to use and allows for most types of number lines to be produced, although it unfortunately doesn’t allow one to produce number lines with fraction symbols such as ½ or ¼. However, I would recommend the image generator as a useful tool that could be used in Mathematics, Science and possibly other subjects as well.

**ACKNOWLEDGEMENT**

The work of the South African Numeracy Chair Project, Rhodes University is supported by the FirstRand Foundation (with the RMB), Anglo American Chairman’s fund, the Department of Science and Technology and the National Research Foundation.

**REFERENCES**


Why Increasing the Number of Compounding Periods Won’t Make You as Rich as You Might Think

Duncan Samson¹ & Craig Pournara²
¹Rhodes University, Grahamstown  ²University of the Witwatersrand
¹d.samson@ru.ac.za  ²craig.pournara@wits.ac.za

INTRODUCTION

Consider the following scenario: R1000 is invested for a single year at 10% p.a. compounded monthly. At the end of the year the balance has accumulated to R1104.71. The total interest earned, i.e. R104.71, is equivalent to 10.47% of the amount invested. Since the power of compound interest lies in the iterative process whereby interest is earned on interest, it makes intuitive sense that if you compound interest more frequently, you will earn more money. Many school textbook tasks suggest that this is the case when they change the frequency of compounding from annual to monthly to weekly. But is this intuition correct?

Let’s investigate this idea by considering our initial scenario, i.e. R1000 invested for a single year at 10% p.a., but where the compounding period is weekly as opposed to monthly. In such a scenario, assuming 52 weeks in a year, the balance in the account at the end of the year would be $1000 \left(1 + \frac{0.1}{52}\right)^{52} = R1105.06$. The total interest earned, i.e. R105.06, is now equivalent to 10.51% of the amount invested. In other words, by increasing the compounding frequency from monthly to weekly we have only managed to increase the interest earned from 10.47% to 10.51% of the initial investment, which seems to be a fairly modest improvement. What if the compounding frequency was increased from weekly to daily, or hourly or even more frequently?

<table>
<thead>
<tr>
<th>Compounding frequency</th>
<th>Calculation</th>
<th>Balance</th>
<th>Interest earned</th>
<th>Interest as % of investment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually</td>
<td>$1000 \left(1 + \frac{0.1}{1}\right)^1$</td>
<td>R1100</td>
<td>R100</td>
<td>10%</td>
</tr>
<tr>
<td>Semi-annually</td>
<td>$1000 \left(1 + \frac{0.1}{2}\right)^2$</td>
<td>R1102.50</td>
<td>R102.50</td>
<td>10.25%</td>
</tr>
<tr>
<td>Quarterly</td>
<td>$1000 \left(1 + \frac{0.1}{4}\right)^4$</td>
<td>R1103.81</td>
<td>R103.81</td>
<td>10.38129%</td>
</tr>
<tr>
<td>Monthly</td>
<td>$1000 \left(1 + \frac{0.1}{12}\right)^{12}$</td>
<td>R1104.71</td>
<td>R104.71</td>
<td>10.47131%</td>
</tr>
<tr>
<td>Weekly</td>
<td>$1000 \left(1 + \frac{0.1}{52}\right)^{52}$</td>
<td>R1105.06</td>
<td>R105.06</td>
<td>10.50648%</td>
</tr>
<tr>
<td>Daily</td>
<td>$1000 \left(1 + \frac{0.1}{365}\right)^{365}$</td>
<td>R1105.16</td>
<td>R105.16</td>
<td>10.51558%</td>
</tr>
<tr>
<td>Hourly</td>
<td>$1000 \left(1 + \frac{0.1}{8760}\right)^{8760}$</td>
<td>R1105.17029</td>
<td>R105.17029</td>
<td>10.51703%</td>
</tr>
<tr>
<td>Per minute</td>
<td>$1000 \left(1 + \frac{0.1}{525600}\right)^{525600}$</td>
<td>R1105.17091</td>
<td>R105.17091</td>
<td>10.51709%</td>
</tr>
<tr>
<td>Per second</td>
<td>$1000 \left(1 + \frac{0.1}{31536000}\right)^{31536000}$</td>
<td>R1105.17092</td>
<td>R105.17092</td>
<td>10.51709%</td>
</tr>
</tbody>
</table>

*Learning and Teaching Mathematics, No. 14, 2013, pp. 26-30*
As we can see from Table 1, while weekly compounding is clearly better than monthly compounding, and while daily compounding is clearly better than weekly compounding, it would appear that the improvement is not as dramatic as one might have expected. In fact, the increased gains associated with more regular compounding seem to slacken off rather quickly.

Table 1 clearly shows how the increased gains, brought about through increasing the compounding frequency, seem to approach a limiting value. In other words, there seems to come a point when increasing the frequency of compounding makes a negligible difference to the final accumulated amount. We have deliberately worked to 5 decimal places in the last 3 rows of Table 1 in order to show that the amount of interest does increase slightly, although this cannot be seen when we round the monetary values to 2 decimal places.

So, in terms of actual rand and cents, there is no difference between compounding hourly, per minute or per second when R1000 is invested at 10% p.a. If we make the principal amount much larger, or if we increase the interest rate, there may be a difference of a few cents between compounding hourly, per minute or per second, but the increase in the amount of interest earned is significantly less than one might have intuitively anticipated.

**The Limiting Value of Compound Growth**

In Table 1 we provide numerical evidence that the increased gains, brought about through increasing the frequency of compounding, does not continue forever, but that it reaches a limiting value as the number of compounding periods per year gets larger and larger. In this section we explore the limiting value of compound growth and show that this value is linked to the number $e$. To do this we will work with the compound growth formula but will choose unrealistic values for the principal amount and the interest rate.

Consider the scenario where R1 is invested for one year with an interest rate of 100% p.a. compounded annually. Clearly by the end of the year the investment would have doubled to R2. If compounding were carried out semi-annually, as opposed to annually, then the balance at the end of the year would be $1 \left(1 + \frac{1}{2}\right)^2 = R2.25$. For quarterly compounding the accumulated amount would be R2.44. For monthly compounding the accumulated amount would be R2.61, while for daily compounding the accumulated amount would be R2.71. Once again we can see that as the compounding frequency increases, the amount accumulated at the end of the year increases, but the extent of that gain is slowing down. So it seems that there is a limit to how much we could grow our investment simply by increasing the frequency of compounding.

If we continue the above calculations for more and more compounding periods per year, we get the values in Table 2. We are essentially investigating the behaviour of a special case of the compound growth formula, namely $A = 1 \left(1 + \frac{1}{n}\right)^n$ where $n$ is the number of compounding periods per year. Note that $\frac{1}{n}$ comes from $\frac{100\%}{\text{No. of compounding periods in 1 year}}$. Table 2 summarises the situation for different values of $n$. 

---

*Learning and Teaching Mathematics, No. 14, 2013, pp. 26-30*
TABLE 2: Investigating the behaviour of a special case of compound growth.

<table>
<thead>
<tr>
<th>No. of compounding periods in 1 year</th>
<th>Accumulated amount at the end of 1 year</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$1 \left(1 + \frac{1}{n}\right)^n$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2.25</td>
</tr>
<tr>
<td>3</td>
<td>2.37037</td>
</tr>
<tr>
<td>4</td>
<td>2.44141</td>
</tr>
<tr>
<td>5</td>
<td>2.48832</td>
</tr>
<tr>
<td>10</td>
<td>2.59374</td>
</tr>
<tr>
<td>15</td>
<td>2.63288</td>
</tr>
<tr>
<td>20</td>
<td>2.65330</td>
</tr>
<tr>
<td>50</td>
<td>2.69159</td>
</tr>
<tr>
<td>100</td>
<td>2.70481</td>
</tr>
<tr>
<td>1000</td>
<td>2.71692</td>
</tr>
<tr>
<td>10000</td>
<td>2.71815</td>
</tr>
<tr>
<td>100000</td>
<td>2.71827</td>
</tr>
<tr>
<td>1000000</td>
<td>2.71828</td>
</tr>
<tr>
<td>10000000</td>
<td>2.71828</td>
</tr>
</tbody>
</table>

By the time $n$ reaches 1000000 it looks as though any further increase in $n$ will hardly make a difference to the outcome, or at least not to any level of significance. This can be seen in the graph in Figure 1. The limiting value of 2.71828 is an approximation of the value $e$.

![Figure 1](image-url)
**PROVING THAT THERE IS A LIMITING VALUE**

We now use the idea of limits to show that there is indeed a limiting value to the expression \((1 + \frac{1}{n})^n\) when \(n\), the number of compounding periods per year, becomes very large. To begin with we take as an established result that the derivative of the natural logarithm \(\ln x\) is \(\frac{1}{x}\). In other words, if \(f(x) = \ln x\) then \(f'(x) = \frac{1}{x}\). Thus, if \(f(x) = \ln x\) then \(f'(1) = 1\). We can now use this fact to express the number \(e\) as a limit.

From the standard definition of the derivative, \(f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}\), we have:

\[
f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1 + x) - f(1)}{x} = \lim_{x \to 0} \frac{\ln(1 + x) - \ln(1)}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1 + x) \quad \text{(since } \ln(1) = 0) \]
\[
= \lim_{x \to 0} \frac{1}{x} \ln(1 + x)^{\frac{1}{x}} \quad \text{(using log laws)}
\]

Since \(\ln\) is a continuous function it follows that we can now re-write this as \(f'(1) = \ln \left( \lim_{x \to 0} (1 + x)^{\frac{1}{x}} \right)\).

And since \(f'(1) = 1\) we now have \(\ln \left( \lim_{x \to 0} (1 + x)^{\frac{1}{x}} \right) = 1\) from which it follows that \(\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e\)

(since \(\ln e = 1\)). Putting \(x = \frac{1}{n}\) and noticing that \(\frac{1}{n} \to 0\) as \(n \to \infty\), we have \(\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e\).

Now that we have shown there is a limiting value when the number of compounding periods increases, we will show how this leads to a new formula for continuous compounding.

**A FORMULA FOR CONTINUOUS GROWTH**

We use the term *continuous compounding* to refer to the idea of gaining interest at every instant. We speak of *discrete compounding* when interest is compounded at discrete points in time such as per year, per month or even per second. In this section we derive a formula for continuous compounding and show how it can be applied. We also compare the results from this new formula with the results from the compound interest formula that works with discrete compounding.

We begin with the compound interest formula: \(A = P \left(1 + \frac{r}{n}\right)^{nt}\) where \(P\) is the principal amount, \(r\) represents the nominal annual interest rate, \(n\) is the number of compounding periods per year, and \(t\) is the number of years.

We can rearrange this as follows by means of some algebraic manipulation:

\[
A = P \left(1 + \frac{1}{\frac{n}{r}}\right)^{\frac{n}{r} \times t}
\]

*Learning and Teaching Mathematics, No. 14, 2013, pp. 26-30*
If we let \( k = \frac{n}{r} \) then we have \( A = P \left( 1 + \frac{1}{k} \right)^{kr} \) and we have already shown that as \( k \to \infty \),
\[
\lim_{k \to \infty} \left( 1 + \frac{1}{k} \right)^k = e,
\]
so our equation for continuous compounding becomes \( A = Pe^{rt} \). We now apply this formula to two problems to illustrate its use:

Example 1: How much will I accumulate if I invest R2000 at 7.5% p.a. compounded continuously for 8 years?

\[
A = 2000e^{0.075 \times 8} = R3644.2376
\]

If we compare this answer to the answer from the compound interest formula with discrete compounding daily for 8 years at the same interest rate (ignoring leap years for the sake of simplicity), then \( A = 2000 \left( 1 + \frac{0.075}{365} \right)^{365 \times 8} = R3644.0130 \). The difference is only 22c, which shows that the two formulae give very similar answers. We leave it up to the reader to investigate the difference between the two formulae for compounding per hour, per minute and per second.

Example 2: How long will it take for an amount to treble if it is compounded continuously at 6% p.a.?

\[
A = Pe^{rt}
\]

\[
3P = Pe^{0.06t}
\]

\[
3 = e^{0.06t}
\]

\[
\ln 3 = \ln (e^{0.06t})
\]

\[
1.0986123 = 0.06t
\]

\[
t = 18.310205
\]

It will thus take approximately 18 years and 4 months.

This formula for continuous growth can be used to model any scenario where there is continuous growth and where the rate of change is proportional to the quantity that is changing, for example population growth or growth of plants.

**REFERENCES**


An Alternative Trig Formula for Solving Triangles

Letuku Moses Makobe
Makwe Senior Secondary School, Mohlarekoma Village, Nebo
makobe.moses@gmail.com

This article presents an interesting trigonometry formula that can be used when solving triangles given two sides and an included angle. When the known dimensions are in SAS configuration one generally uses the cosine rule to first work out the length of the side opposite the given angle. This is then followed by a second step in which the sine rule is used to determine the size of the desired angle. The alternative formula presented here allows one to determine the size of the desired angle in one easy step. The formula is presented below in terms of angle B, the desired angle.

\[
\tan B = \frac{b \sin C}{a - b \cos C}
\]

or

\[
\tan B = \frac{b \sin A}{c - b \cos A}
\]

**Figure 1:** An alternative trig formula for solving triangles.

Let’s begin by proving the formula for an acute-angled triangle ABC. First drop a perpendicular from vertex A as shown in Figure 2.

**Figure 2:** Proving the formula for an acute-angled triangle.

Proof:

\[
\sin C = \frac{h}{b} \quad \therefore \quad h = b \sin C \quad \ldots (1)
\]

\[
\tan B = \frac{h}{x} \quad \therefore \quad h = x \tan B \quad \ldots (2)
\]

From (1) and (2) we have \( b \sin C = x \tan B \) from which it follows that

\[
\tan B = \frac{b \sin C}{x} \quad \ldots (3)
\]
Now, \( \cos C = \frac{a-x}{b} \) \quad \therefore \quad x = a - b \cos C

Substituting this expression for \( x \) into equation (3) gives the formula: \( \tan B = \frac{b \sin C}{a - b \cos C} \)

Let us now consider the proof for the case of an obtuse-angled triangle where the desired angle is the obtuse angle itself. As before, begin by dropping a perpendicular from vertex A as shown in Figure 3.

![Figure 3: Proving the formula for an obtuse-angled triangle.](image)

**Proof:**

\[ \sin C = \frac{h}{b} \quad \therefore \quad h = b \sin C \quad (1) \]

\[ \cos C = \frac{a + x}{b} \quad \therefore \quad x = b \cos C - a \quad (2) \]

Substituting these expressions for \( h \) and \( x \) into \( \tan B_2 = \frac{h}{x} \) yields \( \tan B_2 = \frac{b \sin C}{b \cos C - a} \).

Since \( \tan B_2 = \tan(180^\circ - B_1) = -\tan B_1 \) we have \( -\tan B_1 = \frac{b \sin C}{b \cos C - a} \) from which it follows that \( \tan B_1 = \frac{b \sin C}{a - b \cos C} \).

A similar proof holds when the desired angle is one of the two acute angles in an obtuse-angled triangle.

Let us now consider a typical trigonometry question based on solving triangles.

**Question:** Determine the size of angle B in the following diagram. Take note that the diagram is not drawn to scale.

![Figure 4: Making use of the alternative formula.](image)
Ordinarily, given the configuration of SAS, one would first have to use the cosine rule to determine the length of the side opposite the included angle:

\[(AB)^2 = (8)^2 + (13)^2 - 2(8)(13)\cos{40^\circ}\]

\[\therefore AB = 8,583 \text{ (to three decimal places)}\]

Now that we have all three sides of the triangle we can again use the cosine rule to determine the desired angle:

\[\cos{\hat{B}} = \frac{8^2 + 8,583^2 - 13^2}{2(8)(8,583)}\]

\[\cos{\hat{B}} = -0,2281552997\]

\[\therefore \hat{B} = 103,2^\circ\]

Alternatively, instead of using the cosine rule, we could have made use of the sine rule. However, critical to using the sine rule would be remembering to first determine the smaller of the two unknown angles, i.e. angle A.

\[\sin{A} = \frac{\sin{40^\circ}}{8,583}\]

\[\therefore A = 8,36^\circ\]

\[A = 180^\circ - (40^\circ + 36,8^\circ) = 103,2^\circ\]

However, rather than using either of these multi-step processes, we could simply use the alternative formula and arrive directly at the answer in a single step:

\[\tan{\hat{B}} = \frac{b \cdot \sin{C}}{a - b \cdot \cos{C}}\]

Thus, \[\tan{\hat{B}} = \frac{13 \cdot \sin{40^\circ}}{8 - 13 \cdot \cos{40^\circ}},\] therefore \[\tan{\hat{B}} = -4,26648...,\] from which it follows directly that \[\hat{B} = 103,2^\circ.\]

As shown in the above example, when determining the unknown angles in a triangle given two sides and an included angle, this useful alternative formula offers a quick and more direct route to the solution.
Reflecting on a 2\textsuperscript{nd} Round 2013 SA Mathematics Olympiad Problem

Michael de Villiers

\textit{University of KwaZulu-Natal}

profmd@mweb.co.za

The following problem was used as Question 13 in the 2\textsuperscript{nd} Round of the 2013 Senior South African Mathematics Olympiad (SAMO).

Two tangents are drawn to a circle from a point \(A\), which lies outside the circle as shown in Figure 1. The two tangents touch the circle at points \(B\) and \(C\) respectively. A third tangent intersects \(AB\) in \(P\) and \(AC\) in \(R\), and touches the circle at \(Q\). If \(AB = 20\), find the perimeter of triangle \(APR\).

\textbf{Solution}

If we let \(PQ = x\), then from the ‘tangents from a point to circle’ theorem, \(PB = x\), and \(AP = 20 - x\). Similarly, if we let \(QR = y\), we have \(CR = y\), and \(RA = 20 - y\). So the perimeter of triangle \(APR\) is:

\[AP + PQ + QR + RA = (20 - x) + x + y + (20 - y) = 40\]

Quite surprisingly, and perhaps even counter-intuitively, it turns out that the perimeter of triangle \(APR\) is only dependent on the length of \(AB\) and completely \textit{independent} of not only the position and length of \(PR\), but also the size of the circle. So the problem turns out to be based on an interesting underlying theorem. The reader is now invited to experience this first hand by exploring a dynamic, interactive sketch of this theorem online, by varying \(PR\) and the size of the circle by dragging its centre at:

\[\text{http://dynamicmathematicslearning.com/tangent-perimeter-triangle-theorem.html}\]

Of course, a clever student who has had a lot of experience of Mathematical Olympiads might immediately realize, from the given diagram (Figure 1), that since no lengths of \(PR\) are given to fix it, one may simply move it until \(P\) coincides with \(B\), in which case \(R\) coincides with \(A\). It is then immediately
obvious that the constant perimeter of triangle $\triangle APR$ is $2 \times AB = 40$. (Or alternatively, shrink the circle to zero radius). To prevent the few students who might elegantly reason like this, it was then decided by the Olympiad committee to rather let $PQ = 3$; hence making it perhaps a little easier for more students, but unfortunately somewhat obscuring the underlying theorem.

A nice, direct application of the theorem is demonstrated in Figure 2, which shows two triangles $\triangle ABC$ and $\triangle KLM$ overlapping, and circumscribed around the same circle. If either one or both of these two triangles are rotated, then the perimeters of the shaded triangles $\triangle APU$, $\triangle KQP$, etc. remain constant (provided none of the points $P$, $Q$, $R$, etc. move onto the extensions (outside) of any of the sides of triangles $\triangle ABC$ and $\triangle KLM$).

The reader is once again encouraged to experience this result dynamically by visiting the URL link given earlier.

**Note:** Past papers with solutions from 1997 onwards of the Junior and Senior South African Mathematics Olympiad are available at: http://www.samf.ac.za/QuestionPapers.aspx
A Quick Tool for Tracking Procedural Fluency Progress in Grade 2, 3 and 4 Learners

Debbie Stott

South African Numeracy Chair, Rhodes University, Grahamstown
d.stott@ru.ac.za

INTRODUCTION AND CONTEXT
The South African Numeracy Chair (SANC) project works with fifteen schools in the broader Grahamstown area in the Eastern Cape. Among other things, the SANC project works toward improving numeracy proficiency among learners, basing its notion of numeracy proficiency on Kilpatrick, Swafford & Findell’s (2001) definition of mathematical proficiency. This definition comprises five intertwined and interrelated strands: Conceptual Understanding, Procedural Fluency, Strategic Competence, Adaptive Reasoning and Productive Disposition. As part of the SANC project we run a number of regular after-school maths clubs for learners, and in the club activities we strive to develop numeracy proficiency in the learner participants in each of these five strands.

PROCEDURAL FLUENCY
This article specifically focuses on developing and tracking learner progress in one of the five strands, namely Procedural Fluency. Kilpatrick et al. (2001) describe procedural fluency as “skill in carrying out procedures flexibly, accurately, efficiently and appropriately” (p. 116). Although this article focuses only on the procedural fluency strand, this strand should not be seen in isolation. Rather, the five strands should be seen to complement each other, providing an interwoven conceptualization of numeracy proficiency. This is particularly so in the case of procedural fluency and conceptual understanding. Russell (2000) explains that there is a need to balance both skills and understanding and to make sure the learners develop both procedural competence and understanding where one should strive for a “connection between conceptual understanding and computational proficiency” (NCTM, as cited in Russell, 2000, p. 156). Baroody, Feil & Johnson (2007) point out that one of the reasons that Kilpatrick et al. (2001) recommend that the strands of mathematical proficiency be taught in an interwoven manner is because “linking procedural to conceptual knowledge can make learning facts and procedures easier, provide computational shortcuts, ensure fewer errors, and reduce forgetting (i.e., promote efficiency)” (p. 127).

TRACKING PROGRESS IN PROCEDURAL FLUENCY
There are many ways of developing procedural fluency in young learners, and in the after-school maths clubs we focus on developing procedural fluency both explicitly and implicitly through carefully crafted learner activities. Askew (2010) believes that developing procedural fluency is “best done little and often rather than in less frequent, longer blocks of time” (p. 27). He argues that practice in procedural fluency needs to (a) be simple to set up and carry out, (b) be done little and often, (c) keep learners focused on the mathematics, and (d) help each learner see their own progress (p. 28).

This article presents a series of activities that I developed in collaboration with Mellony Graven. These activities are used in the after-school maths clubs to monitor learners’ procedural fluency progress, specifically with respect to speed and accuracy. The activities are quick and easy to administer and mark, and they allow one to see how quickly learners are answering within the allocated time for each activity as

9 For additional information about these clubs see Graven and Stott (2012) as well as Graven (2011).

Learning and Teaching Mathematics, No. 14, 2013, pp. 36-39
well as how accurately they are answering each activity within that time. The changes in these scores over
time provide a picture of each learner’s progress.

**THE ACTIVITIES**
There are seven different activities to choose from, and learners are given a specified amount of time to attempt each activity. Table 1 describes and provides a sample of each activity along with details of the time and mark allocation for each activity. The activities are all freely available from the South African Numeracy Chair Project website\(^{10}\).

**TABLE 1**: Description, details and sample of each of the seven activities.

<table>
<thead>
<tr>
<th>Activity type</th>
<th>Description &amp; sample</th>
<th>Time allocation</th>
<th>Total marks(^{11})</th>
</tr>
</thead>
</table>
| Add and subtract to 10 | Numbers range up to 10. Use the numbers in the shaded header rows and shaded columns to add/subtract e.g. 2 + 3 = 5 and 10 – 2 = 8 | 1 minute for add
1 minute for subtract | 48                                  |
| Doubbling           | Double the shaded number, e.g. double 4 is 8, double 2 is 4                          | 1 minute                 | 17                   |
| Halving             | Halve the shaded number, e.g. half 4 is 2, half 2 is 1                              | 1 minute                 | 17                   |
| Add/subtract 10     | Add 10 to or subtract 10 from the shaded number, e.g. 5 + 10 = 15, 12 – 10 = 2      | 1 minute                 | 20                   |
| Add/subtract 100    | Add 100 to or subtract 100 from the shaded number, e.g. 5 + 100 = 105, 102 – 100 = 2 | 1 minute                 | 20                   |

\(^{10}\) [http://www.ru.ac.za/sanc/numeracyresources/miscresources](http://www.ru.ac.za/sanc/numeracyresources/miscresources)

\(^{11}\) Note: Each answer is worth 1 mark. This column therefore also represents the number of required responses.
These activities are administered at least once a term in each of the clubs that we work with, and the results are captured in a spreadsheet. With repeated administration of the activities over time, this allows one to track each learner’s progress.

**SCORING THE ACTIVITIES AND EVALUATING PROGRESS**

The results from the seven activities allow one to see how quickly learners are working as well as how accurately they are working within the time limit for each activity. For each activity three different scores are calculated:

- **Actual mark**: The number of items a learner correctly answered.
- **Completion %**: The number of items answered by a learner (either correctly or incorrectly) as a percentage of the total number of items in the activity.
- **Accuracy %**: The number of items a learner correctly answered as a percentage of the number of items answered.

The completion and accuracy percentages allow one to track two things: (a) the speed at which learners are answering in the given time period (a completion rate), and (b) the accuracy of their work in that time (an accuracy rate). While these two rates provide useful information about each learner, they should of course not be interpreted in isolation. A learner who provides answers for every item in a particular activity, but who guesses these answers, would score 100% for completion but very low for accuracy. Similarly, a learner who only answers one item in a particular activity, but who answers that item correctly, would score very low on the completion rating while scoring 100% for accuracy. It is important then to review both the completion and the accuracy scores together in order to get a meaningful picture of a learner’s progress over time.

By way of example let us consider the doubling activity administered to a hypothetical learner in two different terms. The scenario is shown in Table 2.
Table 2: Doubling activity administered to a hypothetical learner in two different terms.

<table>
<thead>
<tr>
<th>Term</th>
<th>Total marks for activity</th>
<th>Actual mark</th>
<th>Completion %</th>
<th>Accuracy %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term 1</td>
<td>17</td>
<td>7 items correctly answered</td>
<td>10 items answered in total</td>
<td>7 out of 10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7 out of 17 = 41%</td>
<td>10 out of 17 = 59%</td>
<td>70%</td>
</tr>
<tr>
<td>Term 2</td>
<td>17</td>
<td>12 items correctly answered</td>
<td>15 items answered in total</td>
<td>12 out of 15</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12 out of 17 = 71%</td>
<td>15 out of 17 = 88%</td>
<td>80%</td>
</tr>
</tbody>
</table>

The doubling activity comprises 17 marks, one mark per item/response. In the first term the learner answered 7 of the 17 items correctly, giving an actual mark of 41%. However, on closer inspection we see that the learner only answered 10 of the 17 items (59% completion) and that of these 10 answers, 7 were correct (70% accuracy). This more nuanced analysis, i.e. the completion and accuracy scores, gives a significantly different perspective to the 41% overall score.

In the second term, when the doubling activity was administered again, the learner answered 12 of the 17 items correctly, giving an actual mark of 71%. With respect to completion and accuracy we see that the learner answered 15 of the 17 items (88% completion) and that of these 15 answers, 12 were correct (80% accuracy). Comparing the second term to the first we can now see that the actual score for the doubling activity increased from 41% to 71%, the completion score increased from 59% to 88%, and the accuracy score increased from 70% to 80%. This provides a rich picture of the learner’s progress.

Suggestions for using the activities
The seven different activities described in Table 1 are available from the SANC website and can be downloaded from http://www.ru.ac.za/sanc/numeracyresources/miscresources. The activities could perhaps effectively be used in Grade 2, 3 and 4 classrooms and could be administered at the beginning of the 2nd and 4th terms. Learners could be involved in marking the activities if required, as well as logging their own scores on a special chart or sheet in order to see their own progress.

Acknowledgement
The work of the South African Numeracy Chair Project, Rhodes University is supported by the FirstRand Foundation (with the RMB), Anglo American Chairman’s fund, the Department of Science and Technology and the National Research Foundation.

References
Timelines for Annuities – Getting to Grips with the Conventions

Craig Pournara
University of the Witwatersrand
craig.pournara@wits.ac.za

Timelines provide a graphical representation that enables us to clarify the timing of transactions – whether payments or withdrawals. They are typically introduced at Grade 11 level, but sometimes the initial problems are so easy there seems little need to represent the information on a timeline. When annuities are learned in Grade 12, the use of timelines becomes more important, and then we need to be clear on the conventions. In this short article I discuss four important conventions of timelines, and I explain why these conventions are necessary. But first a brief comment on modelling the timing of payments.

An annuity is generally defined as a series of equal payments made at regular intervals. In reality these payments can be made at any time of the month. However, in modelling annuity scenarios at school level we consider only two cases: payment at the beginning of the period, and payment at the end of the period. When dealing with payments at the end of a period, it may be confusing to think about compounding interest at the end of the period and also making a payment at the end of the period – which comes first? By convention, a payment made at the end of the period does not earn interest in the period in which it is made.

Let’s now consider the following annuity problem:

You invest R250 at the end of every month for 18 months. The interest rate is 9% p.a. compounded monthly. How much will you have accumulated at the end of the 18-month period?

This future value scenario may be represented on a timeline as shown in Figure 1. The timeline shows 18 payments of R250 made each month from T₁ to T₁₈.

Now suppose we make a minor adaptation to the problem by changing the timing of the payments:

You invest R250 at the beginning of every month for 18 months. The interest rate is 9% p.a. compounded monthly. How much will you have accumulated at the end of the 18-month period?

We can represent this scenario on a timeline as shown in Figure 2. It also shows 18 payments of R250 made each month from T₁ to T₁₈. The only apparent difference between the timelines in Figure 1 and Figure 2 is that the latter timeline shows the end of the 18th month. Note that although there is no payment at the end of the 18th month, interest will still be gained in this last month since it is only at the end of the 18th month when we want to know how much has accumulated.
The problem, however, is that we have used the markers on the two timelines to indicate different points in time. In Figure 1 we imply that each $T_k$ represents the end of a month, e.g. $T_3$ represents the end of month 3. In Figure 2 we imply that each $T_k$ represents the beginning of a month, so $T_3$ represents the beginning of month 3 (which can also be seen as the end of month 2). So our use of $T_n$ is inconsistent even though it may be adequate for our personal use. This brings us to the need for conventions in timelines.

- **CONVENTION 1**: $T_n$ represents the end of period $n$.
- **CONVENTION 2**: $T_0$ indicates the beginning of the first period.

The previous scenarios are represented using these two conventions in Figure 3 and Figure 4 respectively. $P_t$ represents the monthly payment.

In Figure 3, $T_0$ indicates the beginning of the first period. Thereafter there are $n$ payments, each made at the end of $T_1$, $T_2$, $T_3$ and so on to $T_n$. Note that the final payment is made at the end of the $n^{th}$ period, and there is no payment at $T_0$. This scenario represents the future value of an ordinary annuity.

In Figure 4, $T_0$ represents the future value of an annuity due. There are $n$ payments, with the first payment made at $T_0$ and the last payment made at $T_{n-1}$. Based on conventions 1 and 2 we could interpret this timeline as follows: the first payment is made at the beginning of the first month, the second payment is made at the end of the first month, the third payment is made at the end of the second month and so on until the last payment is made at the end of the second last month ($T_{n-1}$). However, this is not what we wanted to represent, which bring us to conventions 3 and 4.
• **CONVENTION 3**: To indicate a series of payments made at the *end* of a period, we show the first payment at $T_1$ and the last payment at $T_n$, and there is no payment at $T_0$.

• **CONVENTION 4**: To indicate a series of payments made at the *beginning* of a period, we show the first payment at $T_0$ and the last payment at $T_{n-1}$, and there is no payment at $T_n$.

These two conventions show the importance of being able to treat any point $T_k$ on the timeline as the end of period $k$ or the beginning of period $k+1$. In order to decide whether to interpret $T_k$ as the end of the $k^{th}$ period or the beginning of the next period it is important that we can see the beginning and end of the timelines. If we can see only the middle portion of the timelines we can’t tell whether payments are made at the beginning or end of the period. To check this, take your left hand and cover the left end of the timelines in Figures 3 and 4. Then take your right hand and cover the right end of both timelines. The portions of each timeline that you can see are identical.

There is, however, a way around this: we can indicate each payment with $P_1, P_2, P_3$ etc. on the timeline, rather than a generic $P_t$. As can be seen in Figure 5, at any time $T_k$ the payment made at that point is $P_{k+1}$, e.g. $P_3$ is made at $T_2$. We read this as payment $P_3$ being made at the beginning of period 3, although $T_2$ is generally taken to represent the end of period 2.

![Figure 5: Using specific subscripts for $T_n$ and $P_n$.](image)

One more issue is worthy of brief discussion, and it relates to the use of month-names on timelines. Consider the two diagrams below.

![Figure 6: Month-names on timeline.](image)  
![Figure 7: Replacing month-names with T-notation.](image)

In Figure 6 month-names are used to indicate four consecutive months, May to August. Based on an “everyday” reading of the timeline, many people tend to interpret the diagram to indicate that payment 1 is made at the beginning of June, payment 2 at the beginning of July and payment 3 at the beginning of August. By contrast, if we merely replace the month-names with T-notation (Figure 7), then by convention we should interpret the scenario as payment 1 being made at the end of period 1, payment 2 at the end of period 2, etc. So the change in labels on the timelines may suggest a shift in the timing of the payments.

We thus need to exercise caution when using month-names on timelines. While they may make the naming of different periods more meaningful, we can easily fall into the trap of thinking they represent the beginning of the month, rather than the end of the month. This, in turn, may cause us to ignore the interest gained in the last month. In Figure 6, for example, we would need to indicate the end of August in order to cover a four-month period. It may be useful to include both T-notation and month-names on
timelines. In this case, the month-name should be written in the gaps between $T_k$ and $T_{k+1}$ so that there is no ambiguity about the beginning and end of each month (Figure 8).

![Figure 8: Avoiding ambiguity by using T-notation and month-names.](image)

I conclude with some reflections and suggestions on the use of timelines based on my experience of teaching financial maths to pre-service secondary maths teachers.

It seems that learners (and university students) tend to lack an appreciation of the value of timelines because their first exposure may have been in situations where the timeline was not necessary to solve the problem. This is similar to using a simple linear equation such as $5 - 3 = 5$ to introduce formal methods for solving equations. Learners simply don’t see the point of learning a new procedure because the answer is obvious using methods they already know. In my experience, it has been more productive to introduce timelines using a complex problem that requires learners to grapple with a large number of payments and/or changes in interest rates. In such situations timelines are helpful to summarise and capture all the important information. Once learners develop an appreciation for timelines as a tool for solving financial maths problems, the conventions can be made explicit using simpler problems. My experience also suggests that the need for the conventions is not immediately obvious to students, and so it may be more productive to wait for the inconsistencies to emerge from students’ intuitive use of timelines.

Nevertheless, I am well aware that I have not convinced all my students of the importance of the conventions and that many still get correct answers despite using their timelines idiosyncratically. But I have also learned that when students make errors in more complex financial maths problems, the source of the problem can frequently be traced to sloppy use of the timeline.

From the perspective of teaching, I believe it is important for teachers to know the conventions of timelines, to appreciate the need for the conventions and to teach the conventions explicitly to learners. But I also know well that there is no guarantee that learners will stick to the conventions!

**ACKNOWLEDGEMENTS**

This work was supported financially by the Thuthuka programme of the National Research Foundation. Any opinions, findings and conclusions or recommendations expressed are those of the author and the NRF does not accept any liability.

**REFERENCES**


Why Still Factorize Algebraic Expressions by Hand?

Michael de Villiers
University of KwaZulu-Natal
profmd@mweb.co.za

"Mathematicians have always dreamed of building machines to reduce the drudgery of routine calculations. The less time you spend calculating, the more time you can spend thinking." - Ian Stewart (2008, p. 345)

This paper presents and briefly discusses an algebraic expression that came up in a proof that could not be factorized by current computer algebra systems, but had to be done by hand using high school techniques.

In De Villiers (2012), the experimental discovery with Sketchpad is described of the following interesting inequality for a parallelo-hexagon – a hexagon ABCDEF with opposite sides equal and parallel (see Figure 1): $AD^2 + BE^2 + CF^2 \leq 4(AB^2 + BC^2 + CD^2)$.

**Figure 1**: Parallelo-hexagon ABCDEF.

**Proof**
Consider Figure 1 where a parallelo-hexagon is placed on a coordinate grid with vertices C, D, A and B having respective coordinates (0, 0); (a, 0); (b, c) and (d, e). It then follows from the symmetric properties of a parallelo-hexagon that the respective coordinates of vertices E and F are $(a + b - d, c - e)$ and $(a + b, c)$. This gives us the following two equations:

$$4(AB^2 + BC^2 + CD^2) = 4[(c-e)^2 + (b-d)^2 + d^2 + e^2 + a^2]$$

$$AD^2 + BE^2 + CF^2 = c^2 + (b-a)^2 + (a+b-2d)^2 + (c-2e)^2 + (a+b)^2 + c^2$$

Expanding and subtracting the second equation from the first gives us:

$$4(AB^2 + BC^2 + CD^2) - (AD^2 + BE^2 + CF^2) = a^2 + b^2 + c^2 + 4d^2 + 4e^2 - 2ab + 4ad - 4bd - 4ce$$

$$= (a-b)^2 + 4d(a-b) + 4d^2 + (c-2e)^2$$

$$= (a-b+2d)^2 + (c-2e)^2,$$

which completes the proof, since the difference of these equations is the sum of two squares, which is always greater than or equal to zero.
**Computer Algebra**

Computer algebra software packages are becoming widespread, with several freeware options currently available. Such computer algebra software has been available on graphing calculators for some time, and more recently has also started appearing on smart phones. This kind of software is strongly challenging the amount of time traditionally spent teaching and drilling learners at school on how to do complicated algebraic manipulation, since most of this routine work can now be done with the mere press of a button. For example, as shown in Figure 2 using the freeware programme *Eigenmath*, one can simply type in an algebraic expression such as $6x^2 + 7x - 3$ or $6x^3 + 5x^2 - 29x - 10$ and immediately factorize the expression by pressing the ‘Factor’ button. In the same way, one can easily expand or multiply out the expression $(4x^2 - x + 2)(3x^2 + 5x - 3)$ simply by pressing the ‘Expand’ button.

As one who does not personally relish tedious algebraic manipulation and usually seeks every opportunity to save time and use available software to perform mundane tasks, I was curious to see how a computer algebra system might be used in the proof of $AD^2 + BE^2 + CF^2 \leq 4(AB^2 + BC^2 + CD^2)$ for the parallelo-hexagon discussed at the start of this article, rather than me doing all the algebra by hand. The freeware programme *Eigenmath* had no problem expanding and simultaneously simplifying the original equation, $4(c-e)^2+(b-d)^2+d^2+e^2+a^2-2ab+4ad-4bd-4ce$, correctly (as shown in Figure 3) to:

$$a^2+b^2+c^2+4d^2+4e^2-2ab+4ad-4bd-4ce.$$ 

Disappointingly, however, the software was not able to group and then factorize the equation into the required form of the sum of two squares, simply returning it, as shown in Figure 3, in the same form as before.
Even the computer algebra system *Mathematica*, as powerful as it is, could not group and simplify the expression further. Readers may wish to try for themselves with the online (scaled down) version of *Mathematica* at: http://www.wolframalpha.com/examples/Algebra.html

One can see why the software had difficulty factorizing this expression, since it is NOT routine at all, and involves a number of steps. It firstly involves the grouping and respective factorization of the quadratic expressions $b^2 - 2ab + a^2$ and $c^2 - 4ce + 4e^2$ into the respective equivalent forms $(a-b)^2$ and $(c-e)^2$ as well as the taking out of the common term $4d$ to factorize $4ad - 4bd$ to $4d(a-b)$. Lastly, it involves recognizing the quadratic expression in $(a-b)$ and $d$ in order to factorize $(a-b)^2 + 4d(a-b) + 4d^2$ to $(a-b + 2d)^2$ and thereby complete the proof.

Even though the factorization procedures individually are routine exercises at more or less Grade 10 level, it does require some ingenuity and ability to ‘see’ and select the appropriate groupings of terms. Actually, for the given proof the original expression was not “factorized”, but written as the sum of two squares. So it’s no wonder the technology wasn’t able to "factorize" the expression - it’s simply not possible in this particular instance. At the moment it therefore seems that computer algebra systems are not yet powerful or ‘intelligent’ enough to handle a non-routine manipulation such as this (even though it can, in the opposite direction, easily “expand” the sum of two squares).

So, this is perhaps an excellent example to demonstrate to one’s high school or undergraduate students that a basic facility in algebraic manipulation *by hand* can still come in handy, even in a modern computational age! Perhaps we also need to identify more examples such as these where computing technology fails, or is of little value, and focus more on them rather than the current practice of focusing only on the drill and exercise of routine skills that can be more efficiently handled by computing technology.

**References**


Investigations afford a wonderful opportunity for learners to explore a diverse array of mathematical ideas in a meaningful and engaging manner. Not only do investigations often lead to unexpected or serendipitous moments of mathematical discovery, but they also provide an opportunity to nurture curiosity and creativity, both of which are important components of a healthy mathematical disposition.

The investigation discussed in this article was presented to a group of secondary school teachers in the context of a teacher enrichment workshop. Teachers worked in groups of three or four and were tasked with generating as many different solutions as they could to the posed problem. The number and variety of solution strategies arising from what at first seemed like a simple starting point surprised many of the participants. In this article I synthesize the different solution methods that were generated in the workshop along with one or two additional solutions.

**THE PROBLEM**

![Figure 1: The problem.](image)

BD and AC are diagonals of rectangle ABCD.

Prove, in as many different ways as possible, that triangles APD and CPD have equal areas.

Before continuing with this article, the reader is encouraged to engage with the problem and to see how many different solution methods they can arrive at.
**Method 1**

Since P is the point of intersection of the diagonals AC and BD, it follows from the properties of a rectangle that AP, BP, CP and DP all have the same length. From this it is clear that triangles APD and CPD are both isosceles. A number of teachers began by taking the provided diagram and physically cutting out triangles APD and CPD. These two triangles were then cut in half (from vertex P perpendicularly to the opposite side) thus generating four small right-angled triangles. Placing these four triangles on top of one another revealed that they were identical, thereby showing that triangles APD and CPD have the same area, since each contains two of the four identical triangles.

![Figure 2: Physically bisecting triangles APD and CPD.](image)

Although this method doesn’t prove the generality of the situation, the physical process of cutting out the triangles led to an intuitive understanding of the reason behind the generality, and this intuitive understanding led to the more formal solution shown in Method 2.

**Method 2**

This method entails drawing in perpendicular bisectors PE and PF for triangles APD and CPD respectively, and then proving that the four generated right-angled triangles are congruent.

![Figure 3: Formally proving congruency of triangles APE, DPE, PCF and PDF.](image)

Since $\triangle CPD$ is isosceles (equal diagonals of rectangle bisect each other), PF bisects $\angle CPD$. Thus $\angle CPF$ equals $\angle DPF$. But $\angle CPF$ also equals $\angle PAE$ (corresponding angles, $PF \parallel AD$) and $\angle DPF$ is equal to $\angle PDE$ (alternate angles, $PF \parallel AD$). Thus $\angle CPF$, $\angle DPF$, $\angle PAE$ and $\angle PDE$ are all equal. We can similarly show that $\angle APE$, $\angle DPE$, $\angle PFC$ and $\angle PDF$ are all equal. Finally, since the diagonals of a rectangle are equal in
length and bisect each other at \( P \), we have \( AP = BP = CP = DP \). It thus follows that triangles \( APE \), \( DPE \), \( PCF \) and \( PDF \) are all congruent (AAS) and consequently that triangles \( CPD \) and \( APD \) have the same area (since each comprises two of the four congruent right-angled triangles).

There are of course other approaches to proving the congruency of triangles \( APE \), \( DPE \), \( PCF \) and \( PDF \). One could make use of the right-angles \( \hat{A} \), \( \hat{D} \), \( \hat{C} \) and \( \hat{F} \) and prove congruency using the case of \( 90^\circ \), hypotenuse, side. Alternatively one could prove the equivalence of sides \( AE \), \( DE \) and \( PF \), and prove congruency using the case of two sides and an included angle (SAS). Working only with sides one could also use a SSS argument to prove congruency.

**Method 3**

This method shows a simple yet elegant solution to the problem. Beginning with any rectangle, divide the rectangle in half both horizontally and vertically, thereby dividing the original rectangle into four smaller rectangles of equal area. Now divide each of these four rectangles in half by drawing in one of the diagonals for each rectangle as shown in Figure 4.

![Figure 4: Dividing a rectangle into eight triangles of equal area.](image)

Using this approach of iterative halving we have now sub-divided the original rectangle into eight triangles of equal area. And since triangles \( CPD \) and \( APD \) each contain two of these smaller triangles, the area of triangles \( CPD \) and \( APD \) are clearly equal. What is elegant about this solution is that focusing on the step-by-step halving of area avoids having to prove that all eight smaller triangles are congruent by instead simply showing that all eight triangles have the same area.

**Method 4**

From vertex D drop a perpendicular to AC and label it \( h \). If we take CP as the base of triangle \( CPD \) then \( h \) is clearly the perpendicular height of the triangle. If we take PA as the base of triangle \( APD \) then \( h \) is perpendicular to the extension of AP and is thus also the height of triangle \( APD \).

![Figure 5: Dropping a perpendicular from vertex D.](image)
Since AP and CP are equal in length, triangles APD and CPD have equal bases as well as equal heights and must thus consequently have equal areas.

\[ \text{Area } \Delta APD = \frac{1}{2} \times AP \times h \]
\[ \text{Area } \Delta CPD = \frac{1}{2} \times PC \times h \]

but \( AP = PC \), \( \therefore \) \( \text{Area } \Delta APD = \text{Area } \Delta CPD \)

**METHOD 5**

This method shows another rather elegant solution to the problem. In triangle APD, drop a perpendicular from vertex P to side AD. In triangle CPD, drop a perpendicular from vertex P to side CD. These two perpendiculars represent the perpendicular heights of triangles APD and CPD respectively, as shown in Figure 6.

![Figure 6: Comparing perpendicular heights to sides.](image)

Now, since P is the midpoint of the diagonals AC and BD, it follows logically that \( h_1 \) is half the length of CD, and that \( h_2 \) is half the length of AD. If we take AD and CD as the bases of triangles APD and CPD respectively, then it becomes clear that the perpendicular height of triangle APD is exactly half the base of triangle CPD, and the perpendicular height of triangle CPD is exactly half the base of triangle APD. From this observation it logically follows that the areas of the two triangles must be equal.

Expressed more formally:

\[ \text{Area } \Delta APD = \frac{1}{2} \times AD \times h_1 = \frac{1}{2} \times AD \times \left( \frac{1}{2} \times CD \right) = \frac{1}{4} \times AD \times CD \quad \left( h_1 = \frac{1}{2} \times CD \right) \]
\[ \text{Area } \Delta CPD = \frac{1}{2} \times CD \times h_2 = \frac{1}{2} \times CD \times \left( \frac{1}{2} \times AD \right) = \frac{1}{4} \times CD \times AD \quad \left( h_2 = \frac{1}{2} \times AD \right) \]

\( \therefore \) \( \text{Area } \Delta APD = \text{Area } \Delta CPD \)
METHOD 6

This method also makes use of the standard formula for the area of a triangle, but focuses on triangles $APD$ and $ACD$. Begin by dropping a perpendicular to $AD$ from vertex $P$. This perpendicular is the height of triangle $APD$, and $CD$ is the height of right-angled triangle $ACD$ as shown in Figure 7.

![Figure 7: Comparing triangles $APD$ and $ACD$.](image)

Notice that $h_1 = \frac{1}{2} h_2$. Thus, $Area \triangle APD = \frac{1}{2} \cdot AD \cdot h_1 = \frac{1}{2} \cdot AD \cdot \left(\frac{1}{2} \cdot h_2\right)$. Rearranging this slightly gives $Area \triangle APD = \frac{1}{2} \cdot \left(\frac{1}{2} \cdot AD \cdot h_2\right)$. But $\frac{1}{2} \cdot AD \cdot h_2 = Area \triangle ACD$, thus $Area \triangle APD = \frac{1}{2} Area \triangle ACD$. It follows that $Area \triangle CPD$ must also equal half the area of triangle $ACD$, and consequently that triangles $APD$ and $CPD$ have the same area.

METHOD 7

This method makes use of the trigonometric area formula, $Area \triangle ABC = \frac{1}{2} ab \sin C$. If we label angle $CPD$ as $\theta$ then $APD = 180^\circ - \theta$.

![Figure 8: Using the area formula $Area \triangle ABC = \frac{1}{2} ab \sin C$.](image)

Now, $Area \triangle CPD = \frac{1}{2} \times CP \times DP \times \sin \theta$

And, $Area \triangle APD = \frac{1}{2} \times AP \times DP \times \sin(180^\circ - \theta) = \frac{1}{2} \times AP \times DP \times \sin \theta$

But $AP = CP$, $\therefore Area \triangle CPD = Area \triangle APD$
METHOD 8

There are a host of other solutions that involve trigonometry. By way of example, consider Figure 9.

![Figure 9: An alternative trigonometric solution.](image)

Letting $P \hat{A} D = \theta$ it follows that $P \hat{D} A = \theta$, $A \hat{P} D = 180^\circ - 2\theta$ and $C \hat{P} D = 2\theta$. Now, using the sine rule in triangle $APD$ we have:

\[
\frac{AD}{\sin (180^\circ - 2\theta)} = \frac{DP}{\sin \theta} \therefore AD = \frac{DP \cdot 2 \sin \theta \cos \theta}{\sin \theta} = 2 \cdot DP \cdot \cos \theta
\]

Using the area rule: $Area \triangle APD = \frac{1}{2} \cdot AP \cdot AD \cdot \sin \theta = \frac{1}{2} \cdot AP \cdot (2 \cdot DP \cdot \cos \theta) \cdot \sin \theta = AP \cdot DP \cdot \cos \theta \cdot \sin \theta$

Now, $Area \triangle CPD = \frac{1}{2} \cdot CP \cdot DP \cdot 2 \sin \theta \cos \theta = CP \cdot DP \cdot \cos \theta \cdot \sin \theta$

Since $AP = CP$ it follows that $Area \triangle APD = Area \triangle CPD$.

METHOD 9

This method involves an elegant bit of dynamic visualization. Take triangle $BPC$ and translate it vertically downwards so that $BC$ lies on $AD$. Now take triangle $BPA$ and translate it horizontally to the right so that $BA$ lies on $CD$. This process is represented diagrammatically in Figure 10.

![Figure 10: A solution method involving dynamic visualisation.](image)

Each of the two translations result in the formation of a rhombus, $PAP'D$ and $PCP''D$ respectively. And since $AP = BP = CP = DP$, the two rhombi have equal side lengths. Closer inspection, for example noticing that the rhombi have equal diagonals ($AD = PP''$ and $CD = PP'$), reveals that the two rhombi are in fact congruent. Since $CD$ cuts rhombus $PCP''D$ in half (vertically) and $AD$ cuts rhombus $PAP'D$ in half (horizontally), it logically follows that since triangles $CPD$ and $APD$ are each exactly half of identical rhombi they must necessarily have the same area.
METHOD 10

This final method also makes use of dynamic visualization. Let’s simplify the original question by thinking of $ABCD$ as a square rather than a general rectangle. If $ABCD$ is a square then it is clear that triangles $APD$ and $CPD$ have equal areas. Let us now take square $ABCD$ and transform it into a rectangle by stretching it horizontally to the right as shown in Figure 11.

When square $ABCD$ is stretched horizontally to the right to form a rectangle, $h_1$ and $b_2$ remain unchanged but $h_2$ and $b_1$ increase. Since $P$, the point of intersection of the diagonals, is always at the centre of the rectangle, it follows that $h_2$ will always be half of $b_1$. In other words, if $b_1$ increases by a factor $k$, then $h_2$ will also increase by a factor $k$. And since $\text{Area } \triangle APD = \frac{1}{2} \cdot b_1 \cdot h_1$, increasing $b_1$ by a factor $k$ will increase the area of triangle $APD$ by the same factor. Similarly for triangle $\triangle CPD$ ($\text{Area } \triangle CPD = \frac{1}{2} \cdot b_2 \cdot h_2$), increasing $h_2$ by a factor $k$ will also increase its area by the same factor. Thus, starting with a square in which triangles $APD$ and $CPD$ have equal areas, stretching the square horizontally to the right increases the areas of the two triangles by the same factor, thus retaining their area equality. A similar argument holds when the square is stretched in the vertical direction.

CONCLUDING COMMENTS

There are no doubt a number of other solutions which haven’t been explored here, but the ten different solution methods presented in this article demonstrate the remarkable mathematical richness that can be drawn from a seemingly simple starting point. What is particularly meaningful about these rich exploratory experiences is that the richness emerges through a process of genuine mathematical discovery and dynamic engagement, thereby nurturing curiosity and creativity.

When carrying out mathematical explorations in the classroom it is important to choose an activity where all learners have a knowledge base that will at least allow them to begin exploring the problem with a certain degree of confidence. It may of course be necessary to provide some level of scaffolding to extend the exploration beyond a basic level, but this should be done in a way that merely prompts learners to explore a new avenue or consider a different aspect of the given problem without compromising their experience of genuine mathematical discovery.

ACKNOWLEDGMENTS

The work of the FRF Mathematics Education Chair, Rhodes University is supported by the FirstRand Foundation Mathematics Education Chairs Initiative of the FirstRand Foundation, Rand Merchant Bank and the Department of Science and Technology.
**ERRATUM**

In the December 2012 issue of LTM an incorrect version of the sine rule appeared in Michael De Villiers's article "Deriving the Composite Angle Formulae for Sine from Ptolemy" (p. 19). The correct version is $a/\sin A = b/\sin B = c/\sin C = 2R$, and not $a/\sin A = b/\sin B = c/\sin C = 1/(2R)$ as accidentally given. However, the web reference (http://www.artofproblemsolving.com/Wiki/index.php/Law_of_Sines) to a proof of this result gave the correct result, as astute readers would have noticed.
AMESA MEMBERSHIP APPLICATION / RENEWAL FORM

Please complete in full and in capital letters

1. Membership No (if renewal): ______________

2. Province: ________________________ Branch: ________________________ (if known)

3. Membership type:  □ Individual  □ Institutional  □ Associate (e.g. full-time University student)

4. Field of Interest:  □ Primary  □ Secondary  □ Tertiary

5. For Individual and Associate members only:
Surname: ________________________ First Name: ________________________ Title: ___________

Postal address: _____________________________________________________________ Postal Code: ___________

Tel/Cell. no: (__) __________ Fax: (______) __________ E-mail: ________________________

5a. Name of School or Institution where you teach: ________________________

6. For Associate members only: I hereby declare that I am a full-time, pre-service student at the following tertiary institution: ________________________ Signature: ________________________

Please include proof of registration at tertiary institution with your application.

7. For Institutional members only:
Designation of person to whom correspondence should be addressed (e.g. The HOD Mathematics / Librarian):

Name of Institution: __________________________________________________________

Postal address: _____________________________________________________________ Postal Code: ________________________

Tel/Cell. no: (__) __________ Fax: (______) __________ E-mail: ________________________

8. Payment: South Africa: Individual – R120.00; Institutional – R340.00; Associate – R30.00; Life – R3 000.00
Other African countries, individual: – ZAR150.00;  Non-African countries – USD65.00
You may pre-pay your subscription at the current rate for up to three years.

Choose one of the following methods of payment (indicate with an X):

□ Cash  R __________

□ I enclose a postal order/cheque for R __________ payable to AMESA.

□ Please debit my credit card account (Visa and Mastercard only) with R __________

Card number: __________ 3 Digits on back: __________

Name on card: __________ Expiry date: __________

Signature: __________ Date: __________

□ Please renew my membership automatically each year by debiting my credit card each year.

This authorisation will remain valid until I or AMESA cancel it in writing.

□ Bank transfer. Bank details are as follows:
Bank: ABSA  Branch Code: 632 005  Account Name: AMESA
Type of Account: Current  Account No: 1640 146601

Note: Please enter your name/membership number in the reference section (bottom right hand corner) of the deposit slip. It is vitally important that you fax or e-mail a copy of the transfer slip and this application form to the number below to ensure that your membership is recorded.

The onus is on you to ensure that we receive the relevant information.

Post the completed application form (with the necessary fee) to: AMESA Membership, P.O. Box 54, WITS, 2050
Only if payment is by credit card (you must sign) or bank transfer may you e-mail or fax the form.
Enquiries: Tel: 011 484-8917  Fax: 011 484-2706 or 0865535042  E-mail: Membership@amesa.org.za  Valid for 2013
Suggestions to writers

What is this journal for?
Learning and Teaching Mathematics is a journal of the Association for Mathematics Education of South Africa (AMESA). This journal aims to provide a medium for stimulating and challenging ideas, offering innovation and practice in all aspects of mathematics teaching and learning. It seeks to inform, enlighten, stimulate, correct, entertain and encourage. Its emphasis is on addressing the challenges that arise in the learning and teaching of mathematics at all levels of education. It presents articles that describe or discuss mathematics teaching and learning from the perspective of a practitioner.

What type of submissions are we calling for?
The types of articles considered for publication in Learning and Teaching Mathematics are:

- **Ideas for teaching and learning**: articles in this section report on classroom activities and good ideas for teaching various mathematics topics. This includes worksheets, activities, investigations etc.
- **Letters to the editors**: discussion pieces that raise important issues on the teaching and learning of mathematics and current curriculum innovations. Views and news on current initiatives
- **Kids say and do the darndest things**: personal anecdotes of something mathematical that has happened in a classroom
- **Window on a Child’s Mind**: description of a classroom event that you want the Journal to respond to.
- **A day in the life of ...**: includes stories about a head of department, a maths teacher, an NGO worker etc.; it could also be an account of a visit to another mathematics classroom... another school... another country...
- **Reviews**: reviews of maths books, school mathematics textbooks, videos and movies, resources including apparatus and technology etc.
- **Webviews**: reviews of mathematics education related websites
- **Help wanted** is a question and answer column: teachers can send their questions on teaching specific topics or aspects to this column for fellow colleagues in the AMESA community to respond to.

What are the technical requirements for the submission of articles?
Articles should not exceed 3 000 words and must be written in English. Articles as short as 300 words are also accepted and of course many of our categories such as “Question and Answers”, “Kids say and do the darndest things”, “Letters to the editors” and so forth can be even shorter. Articles should include the title, author’s name, institution and full postal address, email and contact telephone numbers of the author.

Send your articles by e-mail (in a Word compatible format) to LTM@amesa.org.za. For post or fax submissions use the following address:

Dr Duncan Samson  
Rhodes University  
Education Department  
P.O. Box 94  
Grahamstown  
6140  
Tel: +27 (0)46 603 7210  
Fax: +27 (0)46 603 8084
Provide for their dreams
even when you’re no longer there

Get an Old Mutual Life Plan today and your loved ones can receive from R50 000 up to R500 000 so that they can cope financially.

Other benefits include access to Old Mutual Family Support Services like:
- Health Support
- Trauma, Assault & HIV Treatment
- Funeral Support (Transportation of the deceased)

Leave your family financially secure by SMSing LIFE to 31278 and an Old Mutual Financial Adviser will call you.