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*Learning and Teaching Mathematics, No. 13, 2012, p. 1*
Dear LTM readers

Issue 13 of LTM is once again filled with wonderful ideas that we hope you will enjoy trying out in your own classrooms. This edition contains a wide variety of submissions, both in terms of content, institutional affiliation, and country of origin. We have authors hailing from South Africa, the USA (Hawaii, Texas and Massachusetts), England, Israel, New Zealand and Hong Kong.

We are again delighted that there are a number of articles in this issue that focus on Primary School mathematics – Dorit Patkin and Ilana Levenberg’s exploration of geometry from the world around us, and Paula Arvedson’s use of movement to teach geometry. There are also a number of articles that encourage important aspects of visual reasoning, a critical element in terms of mathematical representation. In addition, there are a number of interesting articles that explore aspects of mathematics and language.

The life of a teacher is incredibly busy, but we would love to have more submissions from practicing teachers. Your article could be as short as one page – e.g. an anecdote or reflection, or a review of a mathematics book, movie or website. Or it could be a longer article about an idea for teaching and learning. Suggestions to authors, as well as a breakdown of the different types of article you could consider, are printed on the last page of this journal. If you have an idea but aren’t sure how to structure it into an article, you’re welcome to email one of the editors directly – we’d be happy to engage with you about turning your idea into a printed article.

Duncan Samson, Marcus Bizony & Lindiwe Tshabalala
The Role of Physical Manipulatives in Teaching and Learning Measurement

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Measurement is a critical aspect of mathematics that affords opportunities for learning while applying and engaging with a host of other mathematical topics (Clements & Bright, 2003, p. xi). Although measurement is a theme that permeates all areas of mathematics as well as day-to-day life, research has shown that many learners find it an aspect of mathematics that is difficult to grasp, with learners often “not understand[ing] the attribute being measured or the units that are used for measurement” (O’Keefe & Bobis, 2008, p. 391). Learners often find particular difficulty in determining the surface area and/or volume of a given object. Van de Walle (2004) argues that when learners are only taught the performance of the skills of a particular procedure at the expense of developing and engaging with the concept itself, they become reluctant to attach meaning to it. This problem poses many challenges for mathematics teachers.

In order to respond to this challenge, many educational researchers advocate the use of physical manipulatives in order to “build connections between mathematics concepts and representations, fostering more precise and richer understanding” (CITEd, n.d.). Physical manipulatives are tangible artefacts that encourage a hands-on engagement with the topic under consideration, and which are specifically designed to foster learning in a teaching and learning environment (Zuckerman, Arida & Resnick, 2005).

In order for my learners to explore surface area and volume in a meaningful hands-on manner I designed a series of four activities that used physical manipulatives to gradually develop learners’ conceptual understanding of these two important concepts.

Activity 1
The first activity focused on determining the area of squares and rectangles. The activity aimed to support learners’ understanding that the term ‘area’ simply refers to the number of squares of a chosen size that cover a particular region/shape or which are enclosed within a specified boundary. Learners were presented with square grid paper which contained square or rectangular bounded regions (Figure 1).

![Figure 1: Grid paper containing square or rectangular bounded regions.](image)
Learners were required to determine the area of each bounded region by identifying the number of unit squares contained within each region. To assist with this process, learners were given square tiles made from laminated paper that they could use to tile the enclosed regions. The emphasis of this process was for learners to share their different strategies for determining the number of unit squares in each region, with particular emphasis being placed on finding the most economical strategy – for example multiplying the number of squares in a single column by the number of identical columns, thereby forging a conceptual link to the conventional formula for determining the area of a rectangle: Area = length × breadth.

Activity 2
This activity entailed designing nets for rectangular prisms, and subsequently constructing the 3-dimensional rectangular prisms from the nets (Figure 2). The various nets were constructed from cardboard on which a square grid had been carefully drawn. When the nets were folded to form the prisms, the folding was done in such a way as to ensure that the grid was on the outside of the shape. This provided a crucial conceptual link to the previous activity, and paved the way for learners to explore the concept of total surface area in Activity 3.

![Figure 2: A rectangular prism formed from a net drawn on gridded cardboard.](image)

Activity 3
This activity focused on determining the area of each of the six faces of the rectangular prisms constructed in Activity 2. In order to accomplish this, learners were encouraged to draw on their experiences from Activity 1 in which the concept of area was explored in terms of the number of unit squares contained within a bounded region. This conceptual link allowed learners to engage with the idea of total surface area, i.e. the number of unit squares on the surface of a given object.

![Figure 3: Learners determining the total surface area of rectangular prisms.](image)
Activity 4
This activity moved the focus from surface area to volume, but was conceptually linked to the previous activities. In Activity 2 the nets that were designed for the rectangular prisms were constructed from cardboard on which a square grid had been carefully drawn. This square grid was drawn such that the size of the unit square contained in the grid exactly matched the dimensions of a large number of identical dice which had been purchased for Activity 4. Each face of the dice thus exactly matched the unit squares on the surface of the rectangular prisms, and the volume of these rectangular prisms could thus be explored by packing the prisms with unit cubes represented by the dice (Figure 4). The main aim of this task was to establish the idea that determining the volume of an object is all about finding the amount of space, expressed in terms of unit cubes, contained within the object.

Figure 4: Determining the volume of a rectangular prism by packing it with unit cubes (dice).

Concluding comments
The four activities described in this article were carefully designed and sequenced to gradually develop learners’ conceptual understanding of surface area and volume. The activities were designed to engage learners in meaningful hands-on tasks through the use of simple physical manipulatives. Although preparation for these tasks was time-consuming, and the process of moving through the sequence of tasks was slow for some learners, I believe the outcome was very encouraging. In the words of one learner, “the use of physical manipulatives has taught us easy methods of calculating the total surface area and volume of prisms … the learning of how to find the total surface area and volume of prisms has been simplified”. The use of simple physical manipulatives in the teaching and learning of area and volume proved successful in (i) providing a tangible context to what is for many learners an abstract concept, (ii) motivating learners through hands-on engagement, (iii) mediating learning by encouraging meaningful and contextualised learner-to-learner and teacher-to-learner conversation, and (iv) allowing learners to explore and experience the fundamental difference between the two important concepts of surface area and volume.

Acknowledgement
This work is based on research conducted under the auspices of the FRF Mathematics Education Chair hosted by Rhodes University, supported by the FirstRand Foundation Mathematics Education Chairs Initiative of the FirstRand Foundation, Rand Merchant Bank and the Department of Science and Technology.

References
It is always nice when an idea for a task comes from a member of the class (Silver, 1994; Kilpatrick, 1987). I had asked Year 7 students to draw several graphs of the form $y = mx + c$, where $m$ and $c$ are constants, choosing for themselves what values to use for the constants. The idea was to look for the effect of $m$ and $c$ on the shape of the graph. Making the link between the form of an equation and the appearance of its graph is something that is reported as being difficult (Knuth, 2000a, 2000b).

Some students varied their $m$ and $c$ values wildly, but most chose small integers and selected them reasonably systematically. One student, Kyle (a pseudonym), began with the following equations: (A) $y = 3x - 2$ and (B) $y = 2x + 1$. He noticed that the lines intersected at an integer lattice point, (3,7) (Figure 1). He began to seek another line that would pass through the same point. He explains in Figure 2 how he did this.

![Figure 1](image-url)
His third graph (C) was \( y = x + 4 \). He was particularly focused on the distance along each line between adjacent points. His approach to describing the slope of his graphs was different from the usual way of comparing the vertical increase for unit horizontal increase (Stump, 2001). Instead, he focused on the distance along the line between points plotted with unit horizontal increase. The spacing of the crosses along his line gave him a measure of slope – the closer together, the shallower the line; the further apart, the steeper the line. (This dynamic approach of running along the line and noticing how frequently the crosses appear perhaps relates to work done in science with ticker-tape, where the machine punches a hole at regular time intervals as the tape is pulled through at varying speeds. Note also that Kyle’s measure of slope corresponds to using the trigonometric function \( \sec \theta \) rather than the more usual \( \tan \theta \).)

When he mentions “the same distance between points”, Kyle explained to me that he meant the vertical distance between points with the same \( x \)-value on different lines. In Figure 1 he has marked ‘5 cm’ vertically twice: once between graphs A and B and once between graphs B and C. Although these are equal, it is easy to sympathise with his comment, “But it doesn’t look it!” There is an optical illusion here, where the eye is drawn to the angle whereas what Kyle is measuring is the vertical distance. This could be seen as relating to the fact that the trigonometrical function \( \tan \theta \) is not a linear function, and this was an interesting outcome from this task that I had not anticipated.

Meanwhile, other students had been drawing various lines and coming to conclusions about the roles of \( m \) and \( c \) in determining the orientation and position of the line \( y = mx + c \). I thought it would be interesting to exploit Kyle’s interest in lines that pass through a common point to stimulate some consolidation work. I took the class to a computer room and presented them with a drawing I had created showing 12 lines all passing through the point \((2;3)\) (Figure 3). I asked them to use graph-drawing software to recreate my picture. This turned out to be an effective way of assessing students’ understanding of the impact of \( m \) and \( c \) on the position and slope of the line. I encouraged those who were having to do a lot of trial and error to refer to the previous task and attend to how the values of \( m \) and \( c \) were changing the lines.

One student describes how she tackled this task in Figure 4. (Although she uses the word ‘equation’ to describe just the right-hand side - where her focus was - she had to enter the ‘\( y = \)’ part as well in order for the computer to draw the line.) It is interesting that she describes the finished picture as a “full circle”, even though (as she must, in some sense, know) it is composed entirely of straight lines. Such an effect is reminiscent of what happens in curve stitching (Millington, 1989).

---

**Figure 2**

I wanted to find out if I could do another line that went through the same point, \((3,7)\). I looked at my tables for the A and B. The difference between \( y \) one \( y \) and the one next to it was 3 on A and 2 on B. So I knew if I started with \((3,7)\) and made the difference either side I then I would get, as shown on graph, the same distance between points. But it doesn’t look it.
We used a graph creating programme on the computer. We were given a piece of paper with lots of lines all coming from or going through the point (2,3).

On the computer I typed in \( y = x + 1 \).

A line was drawn and it went through the point (2,3). Next I experimented until the equation \( 2x - 1 \) drew another line that went through (2,3). Then, \( 3x - 3 \) went through (2,3). Then I noticed the pattern.

You go down the \( x \) numbers in the normal number pattern (e.g., 1, 2, 3, 4, 5, etc.) you then put a plus sign (+) and put a minus number. The minus numbers go down in odd numbers.

So, example,

\[
\begin{align*}
y &= 3x - 3 \\
y &= 4x - 5 \\
y &= 5x - 7 \\
y &= 6x - 9 \\
\text{ etc.}
\end{align*}
\]

You keep going like this until you make the full circle.
It was quick and easy to assess students’ work at a glance as I circulated around the room (Figure 5). All students noticed the patterns in the numbers in their equations and several spontaneously continued beyond the 12 lines given to generate more lines passing through (2;3) (e.g., Figure 6: notice again that this student describes his picture as a ‘cool curve’, although it consists only of straight lines.)
The final task was intended to develop students’ ability to generalise (Mason, 1996). I asked them to make a similar drawing but with all their lines passing through a different common point of their choice. It was easy for students to differentiate this task for themselves (Watson, 1995) by choosing a point with non-integer coordinates (harder) or the origin (easier), depending on what they felt comfortable with attempting. Some tried to alter the equations that they had found for (2,3) to translate the whole thing to another point - with mixed success! Most could now use their knowledge of the significance of $m$ and $c$ to find the lines they wanted much more quickly, and all discovered interesting patterns in their formulae.

Lines passing through the common point $(p;q)$ will satisfy the equation $y - q = m(x - p)$, giving $y = mx + (q - mp)$, so the $c$ value for a given $m$ value must be given by $c = q - mp$. For other ways of using graph-drawing software to create pictures, see Foster (2011).

Making connections from the symbolic/algebraic to the visual/geometrical is vital for students’ subsequent mathematical development. Tasks such as these ones exploit ICT to confront students repeatedly, and instantly, with the visual result of their algebraic input, allowing them to make adjustments and see the consequences directly. A requirement such as that the lines must pass through a particular point, though with no specification regarding the angle, gives students a definite objective (constraint) amid a great deal of flexibility (freedom). In this way, students can explore possibilities without descending into simply producing random lines all over the screen. Having an aim for how the lines should be focuses students’ attention on the details of the image and how to make modifications to the line by altering the equation. Such ways of working enable students to build up strong links between the algebraic and the geometrical and provide a firm foundation for future study.

References


A Numerical Method for Finding the Equation of Any Quadratic Sequence

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There have been a number of articles in past issues of LTM that have focused on various methods for determining the general formula for a given quadratic sequence (see for example Samson, 2008; Bowie & du Plessis, 2009). This article adds to the growing discussion around quadratic sequences.

To begin with, have you ever noticed that multiplying the corresponding terms of two linear sequences together always yields a quadratic sequence? By way of example, consider the two linear sequences 2, 3, 4, 5 and 6, 7, 8, 9. Multiplying corresponding terms yields the quadratic sequence 12, 21, 32, 45 as illustrated below. Apart from anything else, this observation provides an easy method for creating your own quadratic sequences!

<table>
<thead>
<tr>
<th>Row 1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 2</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>Row 1 × Row 2</td>
<td>12</td>
<td>21</td>
<td>32</td>
<td>45</td>
</tr>
</tbody>
</table>

Now, while it is true that the product of any two linear functions is quadratic, not every quadratic function can be expressed as the product of two linear functions. However, any quadratic function of the form \(an^2 + bn\) can always be expressed as \(n(an + b)\) where \(n\) is not only a factor of the expression but the step number (i.e. the independent variable) as well. Thus, in order to create your own quadratic sequence one can simply use the step number (i.e. \(n\)) for Row 1 and any linear sequence for Row 2, as illustrated below.

<table>
<thead>
<tr>
<th>Row 1</th>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 2</td>
<td>(2n + 1)</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>Row 1 × Row 2</td>
<td>(n(2n + 1))</td>
<td>3</td>
<td>10</td>
<td>21</td>
<td>36</td>
</tr>
</tbody>
</table>

Having done a bit of experimenting with creating my own quadratic sequences using this method I noticed that the difference between the first two terms of the quadratic sequence was always equal to the third term of the linear sequence in Row 2. The generality of this observation can be explained by considering the general formula for a quadratic sequence, \(T_n = an^2 + bn + c\), from which we have \(T_1 = a + b + c\), \(T_2 = 4a + 2b + c\) and \(T_3 = 9a + 3b + c\). The difference between the first two terms is \(3a + b\), and since the linear sequence in Row 2 is of the form \(T_n = an + b\), the third term of Row 2 will also be \(3a + b\). This is illustrated in the example below:

<table>
<thead>
<tr>
<th>Row 1</th>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 2</td>
<td>(5n - 3)</td>
<td>2</td>
<td>7</td>
<td>12</td>
<td>17</td>
</tr>
<tr>
<td>Row 1 × Row 2</td>
<td>(n(5n - 3))</td>
<td>2</td>
<td>14</td>
<td>36</td>
<td>68</td>
</tr>
</tbody>
</table>
Up until now I have only dealt with quadratic sequences of the form $T_n = n(an + b)$, but the basic idea can be extended to quadratic sequences of the form $T_n = a(n^2 + bn + c)$ with the aid of an extra row in the table since $an^2 + bn + c = n(an + b) + c$. Building on from our previous table with $c = 2$:

$$\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Row 1} & n & 1 & 2 & 3 & 4 \\
\hline
\text{Row 2} & 5n - 3 & 2 & 7 & 12 & 17 \\
\hline
\text{Row 1} \times \text{Row 2} & n(5n - 3) & 2 & 14 & 36 & 68 \\
\hline
\text{Row 1} \times \text{Row 2} + 2 & n(5n - 3) + 2 & 4 & 16 & 38 & 70 \\
\hline
\end{array}$$

Notice that since the *same constant* has been added to *all the terms* in the third row, the difference between the first two terms of the final quadratic sequence remains unchanged, and is therefore still equal to the third term of Row 2. This suggests a method of working backwards in order to determine the equation of a given quadratic sequence.

**A METHOD FOR DETERMINING THE GENERAL FORMULA OF A QUADRATIC SEQUENCE**

I will illustrate the method by using the following quadratic sequence: 9, 23, 43, 69, 101 …

**Step 1**

Set up a table and complete the 1st and 4th rows:

$$\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Row 1} & & & & & \\
\hline
\text{Row 2} & & & & & \\
\hline
\text{Row 1} \times \text{Row 2} & & & & & \\
\hline
\text{Row 1} \times \text{Row 2} + \text{constant (c)} & 9 & 23 & 43 & 69 & 101 \\
\hline
\end{array}$$

**Step 2**

Calculate the difference between the first 2 terms of the quadratic sequence and write this value as the 3rd term of the linear sequence in Row 2.

$$\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Row 1} & & & & & \\
\hline
\text{Row 2} & & & & & \\
\hline
\text{Row 1} \times \text{Row 2} & & & & & \\
\hline
\text{Row 1} \times \text{Row 2} + \text{constant (c)} & 9 & 23 & 43 & 69 & 101 \\
\hline
\end{array}$$

**Step 3**

Now use the relationships between the rows to fill in the 3rd term of Row 3.

$$\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Row 1} & & & & & \\
\hline
\text{Row 2} & & & & & \\
\hline
\text{Row 1} \times \text{Row 2} & & & & & \\
\hline
\text{Row 1} \times \text{Row 2} + \text{constant (c)} & 9 & 23 & 43 & 69 & 101 \\
\hline
\end{array}$$

*Learning and Teaching Mathematics, No. 13, 2012, pp. 11-13*
Step 4
By comparing Row 3 to Row 4 we now know that the constant \( c \) is 1. This allows us to complete Row 3, from which we are able to complete Row 2.

<table>
<thead>
<tr>
<th>Row 1</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 2</td>
<td>8</td>
<td>11</td>
<td>14</td>
<td>17</td>
<td>20</td>
</tr>
<tr>
<td>Row 1 \times Row 2</td>
<td>8</td>
<td>22</td>
<td>42</td>
<td>68</td>
<td>100</td>
</tr>
<tr>
<td>Row 1 \times Row 2 + constant ( (c) )</td>
<td>9</td>
<td>23</td>
<td>43</td>
<td>69</td>
<td>101</td>
</tr>
</tbody>
</table>

Step 5
Equations for each of the two linear sequences (Row 1 and Row 2) can now be determined by inspection, from which one can build up the general formula for the quadratic sequence (Row 4).

<table>
<thead>
<tr>
<th>Row 1</th>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 2</td>
<td>3n + 5</td>
<td>8</td>
<td>11</td>
<td>14</td>
<td>17</td>
<td>20</td>
</tr>
<tr>
<td>Row 1 \times Row 2</td>
<td>( n(3n + 5) )</td>
<td>8</td>
<td>22</td>
<td>42</td>
<td>68</td>
<td>100</td>
</tr>
<tr>
<td>Row 1 \times Row 2 + constant ( (c) )</td>
<td>( n(3n + 5) + 1 )</td>
<td>9</td>
<td>23</td>
<td>43</td>
<td>69</td>
<td>101</td>
</tr>
</tbody>
</table>

Thus, the general formula for the quadratic sequence 9, 23, 43, 69, 101… is \( n(3n + 5) + 1 \) which, in my opinion, has much more meaning than the simplified expression \( 3n^2 + 5n + 1 \).

Concluding Comments
There are many methods for determining the general formula of a quadratic sequence. Although this particular method isn’t necessarily the easiest, its beauty lies in the simple observation that the product of two linear expressions results in a quadratic expression, and consequently that multiplying the corresponding terms of two linear sequences together yields a quadratic sequence. This allows for an easy method of generating quadratic sequences. In addition, this method allows one to determine the general formula of a quadratic sequence by effectively breaking it down into the product of two simple linear sequences and a constant. The general formula for each of the two linear sequences can easily be determined by inspection, and subsequently built up to generate the general formula for the quadratic sequence.

References
Geometry from the World around Us

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THEORETICAL BACKGROUND

Geometry forms an important component in both elementary and high school curricula. However, it is often perceived as being one of the most complex parts of the curriculum. Students frequently experience a sense of travelling to “an isolated island” where everything is structured in a “logical” or “unusual” way, without any relation to daily life.

There are various theories dealing with the development of pupils’ geometric thinking. One of them is the Van Hiele theory. According to this theory, development in the study of geometry progresses in a hierarchical order through various levels of mastery, where partial mastery at a particular level is crucial but not necessarily sufficient for understanding on a higher level. Students cannot function on a particular level if they have not achieved mastery of previous levels. Therefore, those engaged in teaching geometry must relate to the various levels of thinking among students in the class, striving to overcome the students’ difficulties in learning basic geometric concepts.

Crowley (1987) argues that the type of activities given to learners represents the most meaningful consideration in terms of the development of geometric thinking. His findings indicated that in order to enhance meaningful learning there must be complete compliance between the learners’ level of understanding and the level of assignments with which they have to cope.

At the first stages, geometry studies involve visualisation. Pupils whose visual competence is not sufficiently developed will encounter problems in this subject. Moreover, according to Van Hiele’s levels of thinking, visual competence constitutes an important factor when examining learners already at the first stage. Developing visual competence in the course of teaching aims to increase learners’ mathematical power and promote their ability to solve mathematical problems. Furthermore, emphasizing the aesthetic aspect might improve learners’ comprehension, increase their awareness of the importance attributed to examination and observation and, consequently, change their attitude towards the subject.

A recent study by Walker et al. (2011) investigated whether the development of visualisation skills in non-mathematical contexts may confer an advantage for geometric reasoning. The research findings clearly showed that the development of visual links has importance for, as well as very great impact on, the level of comprehension of geometrical content.

To sum up, various studies reiterate the significance of two key elements for the promotion of geometric thinking, namely visualisation and application.

THE ENVIRONMENT AS A SOURCE OF GEOMETRIC ACTIVITY

During geometry lessons, the use of all types of visual displays, pictures, presentations and movies, which show geometry in the pupils’ environment (both natural as well as man-made), constitutes a bridge between the concrete and the abstract. These means capture the sense of sight, enhance awareness of aesthetics and help learners understand the functions of geometry in our life. In addition to encouraging and recommending the use of visual displays, it is important to emphasize the fact that teachers are not always required to prepare special aids. Every object around us can serve as an illustrative means for the subject. Even a football or an orange can be the beginning of a fascinating lesson. It all depends on the extent of imagination and creativity of teachers wanting to evoke interest and improve their pupils’ understanding of geometry.

The significance of the types of activities presented in this article are corroborated by Hershkovitz, Peled and Littler (2009) by their highlighting of how important it is that pupils receive from their teachers open and varied assignments that promote creativity and incorporate the need for observation and the development of visual competence.

The purpose of this article is to share suggestions and examples of tried and tested activities designed to promote and develop geometric thinking. The activities are based on visual illustrations taken from the learners’ environment. The suggested activities incorporate both natural and man-made examples which attempt to bridge the concrete and the abstract. These activities can serve as an introduction to a studied subject, the core of the studied subject or, alternately, they can be introduced in enrichment lessons or as part of a summary of a chapter.

The pedagogical and didactic functions of the activities are that they:

- Offer interesting and unusual mathematical activities to the pupils.
- Encourage mathematical engagement through experience and inquisitiveness.
- Develop the learners’ ability to cope with problems taken from their daily environment.
- Present the relation between mathematics and other disciplines, such as biology, architecture, etc.
- Reduce anxiety of the subject.
- Create opportunities for mathematical activity for pupils who find the subject difficult.

**Pupils’ experiences with “Geometry around us”**

Within the framework of enrichment advanced studies for gifted pupils with special interest in mathematics, we experimented with the learning materials presented in this article with 56 pupils in the 5th and 6th grades. The pupils experienced numerous activities during several lessons dedicated to various geometric topics (squares, circles etc.). A discourse about the question: “What have you learnt in geometry that you had not known previously?” was conducted after each activity.

One of the first activities introduced to the pupils was a photo of a garden shaped like an acute-angled triangle, the plant beds being concentrated at its vertex. The pupils were asked: “Where should we place a single sprinkler so that it irrigates all the plant beds?” The term “in the middle” came up in the pupils’ answers; only a few of them, though, knew how to relate to the term “center of the circle”. The activity also dealt with gardens shaped like obtuse-angled and right-angled triangles. Following this activity, terms like “center of a circle”, “circumcircle” etc. evolved from being virtual or abstract concepts to real and applicable concepts.

Another activity centred on traffic signs and was performed with the pupils while touring a residential quarter and road junctions. We received the following responses from the pupils:

“I have not thought I could find mathematics on the street”.

“I have never seen any relation between things around my house and the teacher’s explanations in class”.

“At first I did not understand what the teacher meant. How can something outside be connected to the drawing in the textbook?”

After the various activities had been carried out, pupils were asked to develop by themselves similar activities associated with the various topics. At the end of the term we organised an exhibition, presenting photos taken by the pupils (some of them were even taken with their mobile phone camera when they came across a suitable environmental exhibit). We also displayed various questions formulated by the pupils based on the exhibit in the respective photos. Below are several examples of the activities carried out by the pupils.

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EXAMPLES OF ACTIVITIES BASED ON NATURAL AND MAN-MADE ARTEFACTS

1. Vegetation in the desert – “Natural calipers”

The picture shows sandy soil and a desert plant, the ends of its leaves “drawing a circle” by means of the wind. The plant, of course, is the center of the circle and the size of the drawn circles depends on the length of the plant leaves as well as the force of the wind.

Suggested questions:
- Try describing in words the phenomenon presented to you.
- What can you say about the place where the grass grows in the drawn circle?
- Why do some plants draw small circles while others draw big ones?
- Are there other phenomena which form shapes similar to those presented in the picture?
- If you examine two plants which are close to each other, try drawing the circles created by the wind. Specify several options.

The above questions comply with the two first levels of the Van Hiele theory. In the first question, pupils have to identify shapes (1st level) while the other questions relate to the features of the shape (2nd level).

2. Butterflies

Photos of various butterflies were presented to the pupils. The activity facilitates reinforcement and development of the learners’ visual skills, in addition to the inculcation of the concept of symmetry and its meaning in geometry.

Suggested questions:
- Try describing in words the exhibit in the photo.
- Make a note of what is similar and what is different in the butterflies in the photo.
- Try examining the term “symmetry” in relation to the different butterfly shapes.

3. Sites in residential areas

The immediate neighborhood of every pupil, or a picture or photograph from another country, can be a site which invokes mathematical activity. By way of example, the Bahá’í Gardens in Israel (above left) or the Alhambra Palace in Spain (above right) both constitute astonishing geometric shapes. Such pictures

can also lead to questions at the first and second level of Van Hiele’s theory, namely questions relating to the identification of basic shapes and recognizing their features.

4. Man-made symmetry in places of worship
Buildings and places of worship, such as churches, cathedrals and monasteries, abound with interesting geometric patterns. Below is an activity that was carried out at the Carmelite monastery (in Haifa, Israel) which contains some wonderful floor designs.

Suggested activities for the pupils:
- When you enter the monastery please keep quiet and respect the place. Observe the uniquely-tiled flooring.
- How is this flooring different from previous ones which you have seen in the past?
- How many types of symmetry do you notice?
- Draw a straight line between the ends of adjacent leaves. Which polygon do you get?
- Repeat this with the inner leaves of identical colour. Which polygons do you get now? Are they regular polygons?
- Draw a straight line between the ends of every second leaf. Which polygon do you get now? Is it also a regular polygon? What is the relation between previous polygons and this one?
- Is it possible to get a regular polygon with an odd number of sides by drawing a line between the edges of the leaves?

5. Traffic signs – geometrical shapes from the street to the classroom
Pupils have to learn traffic signs as indicators of how to behave on the road. This can be seen as a combination of geometry studies and traffic signs as an international “language”.

Suggested questions:
- Which geometric shapes familiar to you serve as traffic signs?
- What characterizes the triangle-shaped traffic signs?
- Why is the “Stop” sign different from other traffic signs? In what way is it different?
- Look at the “Do not enter” and “Yield” signs. Indicate special features about each of the signs.
- With reference to the “Do not enter” sign, estimate the ratio between the rectangular area of the sign and the area of the entire circle.

6. Car logos

The car industry incorporates many geometric shapes and features into their car logos.

![Car logos](image)

Suggested creative activities for the pupils:

- Create a logo for a company by using three different geometric shapes. The logo can be designed only with a ruler and compass – as part of exercising the topic of geometric constructions.
- Choose a car logo and describe it in words by referring to its component shapes and elements of symmetry.

7. Urban centres

Urban centres with skyscrapers and differently shaped houses can serve for solid geometry activities.

Suggested activities for the pupils:

- Describe in words the skyscrapers in the picture.
- Try estimating their height. Explain how you did it.
- Try estimating the number of windows in the façade of the skyscraper on the right. Explain how you did it.

CONCLUDING COMMENTS

To sum up, geometry studies provide a rich environment for the purpose of developing mathematical thinking, developing logical thinking skills, using intuition and developing spatial orientation and acquaintance with the environment in our daily reality. Mathematical activities such as those suggested in this article enhance acquaintance with and inculcation of mathematical processes and the implementation thereof (Van Hiele, 1999), and improve verbal communication in general and mathematical communication in particular. It is hoped that the activities we have suggested in this article provide contexts for meaningful exploration and learning, while at the same time enhancing mathematical communication and engaging pupils with the beauty of geometrical design.

REFERENCES


Van Hiele, P. M. (1999). Developing geometric thinking through activities that begin with play. Teaching Children Mathematics, 5(6), 310-316.

Deriving the Composite Angle Formulae for Sine from Ptolemy

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Ptolemy of Alexandria (approximately 100-168 AD) not only developed his Planetary theory in his treatise *Almagest*, but also proved the following remarkable theorem, known as Ptolemy’s theorem\(^1\), which he used to compute his table of chords (trigonometric tables) that was in use for over 1000 years: “The sum of the two products of the opposite sides of a cyclic quadrilateral \(ABCD\) is equal to the product of the diagonals; e.g. \(AB CD + AD BC = AC BD\).”

Not only is Ptolemy’s theorem a generalization of the theorem of Pythagoras (which is obtained when \(ABCD\) becomes a rectangle), it also provides a straightforward derivation of the composite angle formulae for the sine function as follows. Consider a cyclic quadrilateral with a diagonal of length 1 as diameter of the circle as shown in the first figure below.

\[ \sin (A + B) = \sin A \cos B + \cos A \sin B \]

Then the lengths of the sides of the cyclic quadrilateral are as shown in the diagram. Since the sine law for any triangle states that \(a/\sin A = b/\sin B = c/\sin C = 1/(2R)\) where \(R\) is the radius of the circumcircle\(^2\), it follows that the length of the remaining diagonal for the first figure above is \(\sin (A + B)\). By applying Ptolemy’s theorem, we obtain: \(\sin A \cos B + \cos A \sin B = 1 \cdot \sin (A + B) = \sin (A + B)\). Similarly, the difference formula can be obtained from the second figure above. These derivations are no more difficult than the usual ones presented in South African mathematics textbooks, but have the added advantage of an interesting historical origin.

\[^1\text{A proof of Ptolemy's theorem can be found at: http://www.cut-the-knot.org/proofs/ptolemy.shtml}\]

\[^2\text{A proof of the relationship between the sine law and the circumradius R can be found at: http://www.artofproblemsolving.com/Wiki/index.php/Law_of_Sines}\]
Searching for a Neat Cubic

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It’s that time of year again: I have to set some calculus questions for school exams. Naturally I want to include a question about a cubic in which pupils would be expected to find the \( x \)-intercepts as well as the turning points. It’s easy if we allow one of the factors of the cubic to be repeated, so that the graph has one of its turning points on the \( x \)-axis, since that way we ensure that the turning points have rational coordinates and that the pupils have a moderately easy task.

I was keen to have a cubic without a turning point on the \( x \)-axis, and yet with one of its roots easily identifiable by the pupils so that there would be no need for a hint about how the expression would factorise. In terms of practicalities this meant that one of the roots needed to be \( \pm 1 \) or \( \pm 2 \) so that it could easily be found by trial and error.

After a diligent (and therefore time-consuming) hunt I came across \( y = x^3 - 5x^2 - 8x + 12 \) which factorises as \( (x + 2)(x - 1)(x - 6) \) and which also has a neat factorisable derivative, \( 3x^2 - 10x - 8 = (3x + 2)(x - 4) \). Unfortunately, this question was in one of the past papers that we would be making available to our pupils for practice purposes, and as such was not exactly suitable as an exam question!

I spent some time trying to invent a new cubic expression by selecting a factorisable quadratic to be the derivative, integrating that and then amending the constant to make one root easy – but whenever I did our pupils for practice purposes, and as such was not exactly suitable as an exam question!

I was keen to have a cubic without a turning point on the \( x \)-axis, so that pupils could identify one of its turning points on the \( x \)-axis, and yet with one of its roots easily identifiable by the pupils so that there would be no need for a hint about how the expression would factorise. In terms of practicalities this meant that one of the roots needed to be \( \pm 1 \) or \( \pm 2 \) so that it could easily be found by trial and error.

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I spent some time trying to invent a new cubic expression by selecting a factorisable quadratic to be the derivative, integrating that and then amending the constant to make one root easy – but whenever I did that the other roots turned out to be irrational. I began to wonder how easy it actually is to come up with such a factorisable cubic whose derivative also factorises neatly. Surely \( y = x^3 - 5x^2 - 8x + 12 \) couldn’t be the only one?

I began my quest in a general way by looking at \( x^3 + ax^2 + bx + c \). It was a disgracefully long time before I realised that once I had one ‘neat’ cubic, any lateral shift of it would also result in a neat cubic. So the cubic I knew about, \( (x + 2)(x - 1)(x - 6) \), had many others associated with it. By way of example, replacing each \( x \) in the original with \( x - 1 \) results in \( (x + 1)(x - 2)(x - 7) \), or one could replace each \( x \) in the original with \( x + 3 \) thereby ending up with \( (x + 5)(x + 2)(x - 3) \).

This meant that the cubic expressions I sought were in groups that were identical to each other except for their horizontal position, and I could thus focus for each group on finding just one. For simplicity’s sake it seemed a good idea to focus on one with 0 as a root. Assume also that we will restrict ourselves to a coefficient of 1 for \( x^3 \).

So now I wanted to find a cubic of the form \( (x + a)x(x - b) \) (on nothing more than a hunch, I supposed things would be easier if I made the middle \( x \)-intercept the zero one), whose turning points are rational. I would then be able to adjust the expression by replacing each \( x \) with \( x - 1 \), or \( x + 2 \) … anything to correspond to a lateral shift.

The multiplied out form of the expression \( (x + a)x(x - b) \) is \( x^3 + (a - b)x^2 - abx \), which has as its derivative \( 3x^2 + 2(a - b)x - ab \). Thus what I needed to do was to choose values for \( a \) and \( b \) so that the second-degree derivative had rational roots.

The discriminant of the derivative expression, \( 3x^2 + 2(a - b)x - ab \), is \( 4(a - b)^2 + 12ab \), which can be written as \( 4[(a + b)^2 + 3b^2/4] \). It thus became clear that I would need \( a + b + b^2 \) to be a perfect square in order for the discriminant itself to be a perfect square, which is the condition for rational roots.

At first sight this did not seem likely, because \( a^2 + ab + b^2 = (a + b/2)^2 + 3b^2/4 \), which one would have thought could not be a perfect square unless \( b = 0 \).

So I checked with the original example, \( y = x^3 - 5x^2 - 8x + 12 = (x + 2)(x - 1)(x - 6) \), which after a slight modification (a lateral shift effected by replacing \( x \) with \( x + 1 \)) becomes \( (x + 3)x(x - 5) \). This has \( a = 3 \) and \( b = 5 \) which gives \( a^2 + ab + b^2 = 49 \) which is indeed a perfect square!
Were there other possibilities? A quick investigation with an Excel spreadsheet revealed several, as shown in the following table:

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>$a^2 + ab + b^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>$7^2$</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>$13^2$</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
<td>$19^2$</td>
</tr>
<tr>
<td>11</td>
<td>24</td>
<td>$31^2$</td>
</tr>
<tr>
<td>7</td>
<td>33</td>
<td>$37^2$</td>
</tr>
<tr>
<td>13</td>
<td>35</td>
<td>$43^2$</td>
</tr>
<tr>
<td>16</td>
<td>39</td>
<td>$49^2$</td>
</tr>
</tbody>
</table>

Of course each of these spawns others, formed by multiplying all components by the same multiplying factor. Thus, from $a = 3$ and $b = 5$ we also get $a = 6$ and $b = 10$ (multiplying each by a factor of 2) which gives $a^2 + ab + b^2 = 14^2$.

Working only with the first of these, i.e. $a = 3$ and $b = 5$, we find the following neat cubics:

<table>
<thead>
<tr>
<th>cubic factorised</th>
<th>cubic expanded</th>
<th>derivative</th>
<th>derivative factorised</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x + 10)(x + 7)(x + 2)$</td>
<td>$x^3 + 19x^2 + 104x + 140$</td>
<td>$3x^2 + 38x + 104$</td>
<td>$(3x + 26)(x + 4)$</td>
</tr>
<tr>
<td>$(x + 9)(x + 6)(x + 1)$</td>
<td>$x^3 + 16x^2 + 69x + 54$</td>
<td>$3x^2 + 32x + 69$</td>
<td>$(3x + 23)(x + 3)$</td>
</tr>
<tr>
<td>$(x + 8)(x + 5)x$</td>
<td>$x^3 + 13x^2 + 40x$</td>
<td>$3x^2 + 26x + 40$</td>
<td>$(3x + 20)(x + 2)$</td>
</tr>
<tr>
<td>$(x + 7)(x + 4)(x − 1)$</td>
<td>$x^3 + 10x^2 + 17x − 28$</td>
<td>$3x^2 + 20x + 17$</td>
<td>$(3x + 17)(x + 1)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$(x − 2)(x − 5)(x − 10)$</td>
<td>$x^3 − 17x^2 + 80x − 100$</td>
<td>$3x^2 − 34x + 80$</td>
<td>$(3x − 10)(x − 8)$</td>
</tr>
</tbody>
</table>

Note how in progressing from one row to the next we simply subtract 1 from each $x$. Clearly the list extends infinitely far (in both directions), but the ones shown here are those that have an $x$-intercept of 0, ±1 or ±2. In fact we immediately also have a whole other list: the ones listed above have gaps between the $x$-intercepts (that is, after all, what the values of $a$ and $b$ really mean) equal to 3 units and then 5 units; reversing the order of these will preserve the neatness, and so we have:

<table>
<thead>
<tr>
<th>cubic factorised</th>
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<td>$3x^2 − 38x + 104$</td>
<td>$(3x − 26)(x − 4)$</td>
</tr>
</tbody>
</table>

In summary, if the gaps between the $x$-intercepts are carefully chosen, the resulting cubic will be ‘neat’. What we haven’t done is see whether this covers all the possibilities – nor have I seen how one might provide a more general solution (rather than an abbreviated list of solutions) to the problem: “What natural numbers $a$ and $b$ have the property that $a^2 + ab + b^2$ is a perfect square?” My experiments with Excel suggest that we can have $a^2 + ab + b^2 = q^2$ if and only if $d$ has the form $6k + 1$ and $k$ is not a perfect square – but this is a problem for the number theorists to play with. In the meantime I have enough neat cubics to last me the whole of the rest of my exam-setting career!
Exploring Language Issues in Multilingual Classrooms

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INTRODUCTION
Multilingualism is rapidly becoming a serious challenge for many schools in South Africa, perhaps most noticeably in the Gauteng province. Not only do many schools have learners with a variety of South African indigenous languages as their home language, but numerous schools also have learners from other African countries. As a result of this language complexity many schools have opted for English as the language of teaching and learning despite that fact that many teachers and learners are not fluent in English. As such, the teaching context in such schools is highly complex, and is heavily affected by multilingualism. During the course of learning Mathematics numerous language issues emerge which raise a number of critical questions. How does the fact that teachers teach mathematics in a language that is not necessarily the learners’ home language affect the teaching and learning of mathematics? What impact does the teacher’s home language have on the promotion of learners’ conceptual understanding in the Mathematics classroom?

WHY LANGUAGE AND CONCEPTUAL UNDERSTANDING?
In order to develop mathematical thinking, learners have to be able to communicate mathematically (Setati, 2005). Language issues arise when the language used inside the classroom is different to the language used outside the classroom (Cummins, 2000). To ensure effective verbal communication in the Mathematics classroom, mathematics teachers need to encourage fluency in both oral and written mathematical language (Pimm, 1981). However, a critical aspect of language use in the Mathematics classroom is that ordinary or everyday words are often used with a specialised mathematical meaning, and particularly in a multilingual setting specialised usage of everyday words can often lead to confusion. This raises a crucial question: “how does the use of language in a multilingual class enable or constrain the use of this specialised language of mathematics?” (Pimm, 1981, p. 2).

CLASSROOM VIGNETTES
This article presents and discusses a series of classroom vignettes in order to illustrate the complexity of language issues within the context of a multilingual mathematics classroom. The setting for the vignettes is a Grade 4 classroom where the vast majority of learners have Setswana as home language. The class is being taught Mathematics in English by a teacher whose main language is isiZulu.

VIGNETTE 1
Learners were presented with the diagram shown in Figure 1.
TEACHER: Okay, let’s see, what is the name of this shape?

MPHO: It is a rectangle ma’am.

TEACHER: Why do you say it’s a rectangle? [Learners are quiet and seem to be thinking] Okay, what do you see here? Tell me, how can you explain to someone else that this is a rectangle?

THABO: It has four parts.

TEACHER: Four parts?

THABO: Yes ma’am, the two parts here are short and the other two parts are long.

TEACHER: Come and show us.

THABO: One, two, three, four, ma’am. [Pointing at the sides of the rectangle]

TEACHER: Oh! Thabo, we don’t call that parts, we call them sides. We talk about parts in fractions but now we talk about sides because these are shapes. Do you understand?

LEARNERS: [Silent and looking at the diagram]

TEACHER: Why are you quiet? Don’t you understand? Niyabona (can you see this), shebang mo (look here), side and part are not the same, dintho tse aditswani (these things are not the same). Hmm... [scratching her head], okay remember fractions have parts and shapes have sides. Ka Setswana e yi, okay niyasazi isiZulu nonke ne (okay, you all know Zulu)?

LEARNERS: Yes.

TEACHER: Lame part e shope (these parts of a shape) we call them sides and then [drawing a diagram with four equal parts] lama piece ale diagram wona we call them parts (these pieces of the diagram are called ‘parts’). [The teacher keeps quiet for some time as though deep in thought]

When the communication with the learners started to become challenging the teacher seemed to lack the appropriate mathematical language that was required. When asked to comment on how she had handled the multilingual challenges represented by the above transcript the teacher responded as follows:

I realised that hmm... I also did not know how to explain the difference. I did not know how to explain in Setswana, Zulu or English. I realised that in our everyday language in the township we use the word ‘part’ for almost many things i.e. role, side, pieces of a whole, located place. When I wanted to use Setswana I could not explain further.

This suggests that the teacher realised that she was deficient in the home language of the learners as well as the mathematical language. When asked why she eventually resorted to using isiZulu as opposed to the learners’ home language of Setswana to try to explain the distinction between ‘part’ and ‘side’ the teacher commented that she wanted to feel comfortable with the language she was using in order to have sufficient vocabulary to explain the difference between the two words.

Mathematics teachers often end up spending a lot of their time teaching language in the Mathematics class. One of the dangers that this vignette highlights is how an oversimplification of language can lead to confusion when it comes to the use of proper mathematical terminology. The teacher therefore has a responsibility to ensure that learners understand not only the mathematics, but the language of learning and teaching as well.
VIGNETTE 2

Learners were presented with the diagram shown in Figure 2.

**Figure 2:** A rectangular prism.

**TEACHER:** Okay, let’s look at the diagram. Can you see that it is different from the ones we worked with i.e. square and rectangle? By the way what do we call this one?

**LEARNER:** Prism.

**TEACHER:** What, class?

**CLASS:** Prism.

**TEACHER:** Let’s see the parts of this prism. Remember I told you the names of all these parts of this prism.

Learners suggested the following: ‘corner’ for part A, ‘corner’ for part B, and ‘face’ for part C.

**TEACHER:** This is incorrect. You should not call these parts corners [referring to parts A and B] in this diagram.

**LEARNER:** But it has a corner ma’am. It also looks like the corner of L, like in a rectangle.

**TEACHER:** No, we don’t say that here, you see these corners are not the same as the ones you see in the rectangle.

Learners could not differentiate between the vertex and edge. Instead they made use of the term ‘corner’. The teacher had made use of the term ‘corner’ while discussing 2D objects such as the rectangle. She had also described adjacent sides in the rectangle as forming an ‘L shape’. As a result, the learners associated parts A and B of Figure 2 with the term ‘corner’ because of the teacher’s explanation of the ‘L shape’ in the rectangle. The teacher thought that if she used everyday language, i.e. words that her learners were familiar with, it would make it easier for them to understand. However, not using correct mathematical language ultimately led to confusion in the minds of the learners.

VIGNETTE 3

This short vignette gives an example of how misunderstandings and misconceptions can easily arise in a multilingual setting where specialised mathematical language is internalised in terms of everyday usage.

Having taught the class expanded notation the teacher gave the learners a short assessment. She asked the learners to write 127 in expanded notation. Expecting the answer of 100 + 20 + 7 she was surprised to see one learner give an answer of 100 + 18 + 9, which she duly marked as being incorrect. The learner who had written the answer objected to it having been marked as incorrect: “Ma’am, you said we must make it long because expand means long. I am right, ma’am. If I add all of them it’s 127 which is short.”

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DISCUSSION AND RECOMMENDATIONS

The fact that most teaching and learning in these multilingual classrooms take place in English, which is not the main language of either the teachers or the learners, makes participation more difficult for the learners and the development of ideas more difficult for teachers (Brodie, 2005). This is supported by Schiffer (2001) who argues that before the teacher gives a task to the learners he should consider how he can attend to the mathematics in what the learners will be saying and doing. It is imperative for the teacher to know how he will use the correct mathematical language to ensure that mathematics does not get lost in the process of grappling with the language itself.

At school, language becomes more cognitively demanding. New ideas, concepts and language are presented to the learners at the same time. Setati (2005) argues that in a mathematics class, teachers find themselves having to explain concepts in English first before they can get to the mathematical language. However, teachers should ensure that they use correct mathematical language to assess the mathematical validity of learners’ ideas. Teachers should be able to encourage learners to use correct mathematical language and avoid oversimplification through the use of everyday English language.

Pimm (1981) argues that many children’s difficulty with mathematics may be due to the complexity of language rather than the mathematical task itself. This suggests that English language can be a barrier in the understanding of mathematics in a multilingual classroom where English is a second language not only for the learners but for the teacher as well. However, while the process of grappling with a language can interfere with learning, it can also nurture the learning process (Margaret, as cited in Carpenter, Franke & Levi, 2003). Teachers need to take the responsibility of ensuring that learners are able to distinguish between everyday usage of words and their use in a specialised mathematical context.

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Using Movement to Teach Geometry

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Learners who are physically involved in their mathematics instruction delight in discovering and engaging with concepts as a result of the real-world connections which are forged through mind and body working together. They gain a depth of understanding far greater than any worksheet on its own can ever provide. Their mathematics vocabulary increases and they develop problem-solving skills that can be used in all subject areas. Sack and van Niekerk (2009) assert that “children should develop competence using physical and mental processes with visual representation modes in addition to verbal descriptions, regardless of the representation given in any particular problem” (p. 142). What follows is an example of a Grade 1 geometry unit on identifying and describing two- and three-dimensional shapes which incorporates body movement combined with verbal descriptions.

The purposes of the unit are to (i) help children identify and describe two- and three-dimensional shapes in the real world, (ii) graphically represent the shapes found in the real world, and (iii) demonstrate a perimeter with their bodies. This unit builds on previous work with two-dimensional shapes including the square, triangle, circle, rectangle and rhombus, as well as what the children are beginning to learn about three-dimensional objects such as the cube, sphere, cylinder, rectangular prism, cone and pyramid. The unit is sub-divided into four different activities, Make Your Own, Shapes Treasure Hunt, Shape Walk and Class Shapes Book.

MAKE YOUR OWN

The unit begins with the Make Your Own creative learning centre. To prepare for this activity you will need to collect a large assortment of objects of various shapes and sizes such as empty paper towel holders, empty toilet roll holders, plastic or metal funnels from kitchens or garages (cleaned, of course), empty cereal boxes, empty milk cartons of all sizes, light-weight balls, and so on. Learners can help with this preparation by each bringing an assortment of objects to school. Once they have been collected, the three-dimensional shapes need to be sorted into separate boxes, each one labeled with the shape's name. The learners then use the shapes to work on construction projects of their own choosing. They begin by drawing a plan, which can be a simple sketch, and gather the needed materials. They then carry out their plan and keep a daily journal of progress and challenges. Children can be as creative as they want to be. For example, when this activity was carried out with a class of Grade 1 learners, one child created a miniature town with houses, shops, a bank, two "golden arches" (representing a fast-food outlet) and even a grain silo to hold feed for the imaginary animals. Another child constructed a doll, using a sphere, cylinders, and a rectangular prism. Throughout the activity geometric vocabulary and terminology is used frequently and in context. In addition to the mathematics, this activity affords easy integration of other subjects such as science and art.

SHAPES TREASURE HUNT

At the Shapes Treasure Hunt centre each learner takes a Treasure Hunt check sheet and marks off each time they find a two-dimensional or three-dimensional shape or object, indicating where they found it and how it was being used. This activity can be accomplished in a single lesson, or it could be carried out over a period of several days. Children have the potential to discover shapes on their clothing, their bed sheets, their living room curtains, and objects in their homes, stores, neighborhood and school playground. The awareness of geometric shapes and objects goes far beyond the classroom.
SHAPE WALK

The two previous activities pave the way for the important *Shape Walk* activity. This requires a certain amount of preparation as the teacher needs to carefully plan the walk's course so that the children are able to discover and identify a wide variety of shapes and objects. When the children carry out the *Shape Walk* they should take their journals and pencils with them and make notes or pictures of what they see along the way, having been instructed to "*look up; look down; look all around.*" Part of the walk must include both a rectangle and a circle on the ground that the children can walk on. The teacher helps the children get a "body feel" for the concept of perimeter by having the children hold hands and walk around the perimeter of the object. The teacher has the smallest children measure the perimeter by standing shoulder to shoulder around the shape, making frequent reference to the word *perimeter*. The teacher then has the largest children repeat the measurement, again standing shoulder to shoulder. If the two measurements are different then this provides an excellent opportunity to discuss the importance of standardized measurements. The children who have not yet been involved can then use a tape measure to measure the perimeter in centimetres. Have the children dance (to music, if available) around the shape, then stop and measure the same shape again. Ask if the size of the perimeter has changed when there is a different point of origin. Let the children take turns verifying the size of the perimeter for other objects on the ground. It is helpful to have a recorder to capture the children's discussions as you will find them rich in mathematics vocabulary. Before returning to the classroom give the children time to write and draw in their journals, along with reflections on their movement and their learning.

CLASS SHAPES BOOK

The conclusion of this unit involves making a *Class Shapes Book*. The children choose from the drawings they have made and write a caption for each one. This is their record of their discoveries. They may also include a tally chart or bar graph of the class’s results as a whole, for example a bar graph showing the most frequently occurring shapes in the environment.

ASSESSMENT

Assessment for this unit comes from observations of the children during each of the activities, and during small group and whole class discussions. The teacher needs regularly to inspect the learners’ drawings and journals, not only to get a sense of each child’s progress but in order to address any misconceptions or misunderstandings that may have arisen. These can then be addressed either in small group or whole class discussions.

CONCLUDING COMMENTS

Van de Walle comments that "what a child 'knows' in geometry is not just a list of mastered terms, but a way of thinking in geometric contexts" (p. 349). The goal is to help learners gain that *way of thinking*. Movement in geometry helps achieve that goal since learners internalize their new knowledge because they *experience* it; they *live* it; their bodies *feel* it.

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ACKNOWLEDGEMENTS

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Viviani’s theorem – A Geometrical Diversion

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Viviani’s theorem, named after the Italian mathematician and geometer Vincenzo Viviani (1622 – 1703), is a simple yet delightfully intriguing geometry theorem with a number of interesting extensions. Viviani’s theorem states that for any point inside an equilateral triangle the sum of the perpendiculars from that point to the sides of the triangle is equal to the length of the triangle’s altitude.

![Viviani's theorem diagram](image)

**Figure 1**: Viviani’s theorem: \( d + e + f = h \).

In Figure 1, irrespective of where point \( P \) is placed inside equilateral triangle \( ABC \), the sum \( d + e + f \) will be a constant value. This can be beautifully illustrated using Dynamic Geometry Software (DGS) where one can drag point \( P \) to different positions and see that the sum of the three perpendiculars remains constant. Furthermore, not only is this sum a constant value, but that constant value is the same as the perpendicular height (or altitude) of the triangle: \( d + e + f = h \). Since the triangle is equilateral, the altitude from all three vertices is of course the same. As an interesting aside, it is worth noting that constructing the three perpendiculars from point \( P \) subdivides \( \Delta ABC \) into three cyclic quadrilaterals.

![Proving Viviani's theorem](image)

**Figure 2**: Proving Viviani’s theorem.

Proving Viviani’s theorem is a relatively simple matter. Firstly, join point \( P \) to each of the three vertices, thus forming lines \( PA \), \( PB \) and \( PC \). The original equilateral triangle \( ABC \) has now been subdivided into three smaller triangles: \( APB \), \( BPC \) and \( APC \). The sum of the areas of these three triangles is clearly the same as the area of the original equilateral triangle. This gives

\[
\frac{1}{2}(BC)(d) + \frac{1}{2}(AC)(e) + \frac{1}{2}(AB)(f) = \frac{1}{2}(BC)(h)
\]

where \( h \) is the altitude of the original equilateral triangle. Since \( \Delta ABC \) is equilateral, \( AB = BC = AC \), from which it follows that \( d + e + f = h \).
There is another rather elegant way to prove Viviani’s theorem which I first came across as a proof without words (Kawasaki, 2005). The proof shown in Figure 3 is a slight modification of Kawasaki’s (2005) original proof without words.

![Figure 3](image)

**Figure 3:** Viviani’s theorem – a proof without words.

The proof proceeds as follows. Having chosen an arbitrary point \( P \) inside the equilateral triangle, draw in the perpendiculars from that point to the sides of the triangle. Now create three new equilateral triangles by drawing in three lines passing through the chosen point \( P \) such that each line is parallel to one of the three sides of the original equilateral triangle. The perpendiculars from the chosen point \( P \) will be the altitudes of these three new equilateral triangles (Figure 3a). Next rotate the triangles so that the respective altitudes are vertically oriented (Figure 3b). Slide the leftmost of the three triangles along the edge of the original equilateral triangle until their upper vertices coincide (Figure 3c). Finally slide the bottom two triangles horizontally to complete the proof that the sum of the three perpendiculars is equal to the altitude of the original equilateral triangle (Figure 3d).

The dynamic nature of this proof is incredibly powerful. When I workshop this particular proof I usually prepare an A4 sheet with a large equilateral triangle printed on it. Each learner is then given two copies of the equilateral triangle, one on white paper and the other on brightly coloured paper. Learners then line the two sheets up so that the two equilateral triangles overlap, and mark the point \( P \) using a pair of dividers (or something similar). This ensures that the chosen point \( P \) is in exactly the same place in both triangles. Learners then make a duplicate copy of their construction (Figure 3a), one on the white paper and the other on the brightly coloured sheet. The three smaller equilateral triangles formed are then cut out from the coloured sheet and placed over their matching positions on the white paper. The dynamic nature of the proof can now be enjoyed to maximum effect, since the three equilateral triangles can be rotated (Figure 3b), translated (Figure 3c), and finally slid horizontally (Figure 3d). The final denouement of the proof, when the three triangles line up vertically as in Figure 3d, never ceases to amaze and delight!

To recap, Viviani’s theorem states that for any point inside an equilateral triangle the sum of the perpendiculars from that point to the sides of the triangle is not only invariant, but equal in length to the triangle’s altitude. This equality still holds even if the chosen point lies on one of the three sides of the equilateral triangle (although in such cases one of the perpendiculars collapses to zero). But what if the chosen point \( P \) lies outside the original triangle? There is still an interesting relationship between the three perpendiculars from \( P \) and the altitude of the original triangle, but the relationship changes depending on the exact position of \( P \). If we extend the three sides of the original triangle, then the space outside the triangle is effectively subdivided into six distinct regions (Figure 4).

![Figure 4](image)

**Figure 4:** Extending the three sides.
Let’s choose a point \( P \) in region \( V \) and construct perpendiculars from point \( P \) to each of the three sides as before. Since \( P \) lies outside the original equilateral triangle the perpendiculars are constructed to the extensions of two of the sides (Figure 5a). Next join \( P \) to each of the three vertices as before, thus forming lines \( PA \), \( PB \) and \( PC \) (Figure 5b). We have now created three new triangles: \( \triangle APB \), \( \triangle BPC \) and \( \triangle APC \).

![Figure 5: Choosing a point outside \( \triangle ABC \).](image)

In triangle \( \triangle APB \), if we take \( AB \) as the base of the triangle, then \( f \) is the perpendicular height. The area of \( \triangle APB \) is thus \( \frac{1}{2}(AB)(f) \). Similarly, the areas of \( \triangle BPC \) and \( \triangle APC \) are \( \frac{1}{2}(BC)(d) \) and \( \frac{1}{2}(AC)(e) \) respectively. Comparing these three areas to that of the original equilateral triangle reveals the following relationship: \( \text{Area } \triangle APB + \text{Area } \triangle BPC - \text{Area } \triangle APC = \text{Area } \triangle ABC \). Using the above expressions for the areas of the various triangles we thus have \( \frac{1}{2}(AB)(f) + \frac{1}{2}(BC)(d) - \frac{1}{2}(AC)(e) = \frac{1}{2}(BC)(h) \) where \( h \) is the altitude of the original equilateral triangle. Since \( \triangle ABC \) is equilateral, \( AB = BC = AC \), from which it follows that \( f + d - e = h \). Similar but different expressions can be derived for each of the other five regions. Alternatively, instead of considering each region separately, one can introduce the concept of directed distances, and thus view heights that fall completely outside the triangle as being negative (thus resulting in negative areas), thereby ensuring that the sum \( d + e + f \) remains constant irrespective of whether \( P \) lies inside or outside the original equilateral triangle (De Villiers, 2003, p. 149).

Let’s leave Viviani’s theorem for a moment and make a slight diversion. Take a regular hexagon and place a point \( P \) anywhere inside the hexagon. Next join point \( P \) to each of the six vertices of the hexagon (Figure 6a) thereby splitting the original hexagon into six triangles.

![Figure 6: Slicing up a regular hexagon.](image)
An interesting result of this construction is that if each triangle is paired with the triangle opposite it (Figure 6b) then each pair of triangles will have the same constant area, irrespective of the initial position of point \( P \). In an analogous situation, imagine a cake in the shape of a regular hexagon. Place a knife at a random point \( P \) on the cake and make cuts from \( P \) to each corner. If three people now each take a slice of cake along with the slice diametrically opposite it, they will all end up with the same amount of cake irrespective of where the knife was initially placed. The reason for this can readily be seen from Figure 6c. The pair of triangles highlighted in Figure 6c is \( \triangle APF \) and \( \triangle CPD \). The altitudes from point \( P \) have been drawn in for each of the two triangles. The area of \( \triangle APF \) plus the area of \( \triangle CPD \) can be expressed as \( \frac{1}{2}(AF)(h_1) + \frac{1}{2}(CD)(h_2) \). Similar area expressions can be obtained for the other two pairs of triangles. Now, since the original hexagon is *regular*, all six sides are equal in length, opposite sides are parallel, and the perpendicular distance between opposite sides is constant. From these properties it should be clear why each pair of opposite triangles will have the same area.

Returning to our original slicing of the hexagon (Figure 6a), another interesting property of this slicing is that the six triangles formed can be grouped into two groups of three so that each group has the same total area (Figure 7). If we use the cake analogy again, if two people each take alternate slices then they will each end up with the same amount of cake - a somewhat surprising result.

At this point you should be wondering what all this has to do with Viviani’s theorem. The connection is rather beautiful, and is based on the fact that if one takes a regular hexagon and extends each of the six sides in both directions, each extension being the same length as the sides of the original hexagon, then one creates two identical overlapping equilateral triangles which bound the sides of the hexagon (Figure 8). The six perpendiculars from \( P \) have been drawn in and are indicated as \( h_1 \) to \( h_6 \).

![Figure 7: Two groups of three.](image1)

![Figure 8: Viviani’s theorem strikes again.](image2)
Since the hexagon is regular, let us for the sake of simplicity refer to the length of each side of the nested hexagon as $s$. The sum of the areas of the three shaded triangles highlighted in Figure 7 is thus $\frac{1}{2}(s)(h_1) + \frac{1}{2}(s)(h_3) + \frac{1}{2}(s)(h_5)$ which in factorised form is $\frac{1}{2}(s)(h_1 + h_3 + h_5)$, and from Viviani’s theorem we know that this equals $\frac{1}{2}(s)(H)$ where $H$ is the height of the large equilateral triangle. Using a similar argument, the sum of the areas of the three un-shaded triangles is $\frac{1}{2}(s)(h_2) + \frac{1}{2}(s)(h_4) + \frac{1}{2}(s)(h_6)$ which similarly is also equal to $\frac{1}{2}(s)(H)$. And that rather beautifully completes the explanation.

A final observation with regard to Figure 8 is that the six triangles formed around the perimeter of the original hexagon are all identical equilateral triangles whose sides are the same length as the sides of the original regular hexagon. It thus follows that (i) the sum of the areas of these six triangles equals the area of the enclosed hexagon, (ii) the area of the hexagon is two thirds of the area of one of the large equilateral triangles, and (iii) the area of the three shaded triangles not only equals the area of the three un-shaded triangles (Figure 7) but also equals the area of three of the smaller identical equilateral triangles around the perimeter of the enclosed hexagon (Figure 8). It is left to the reader to confirm the veracity of these last three observations.

Concluding comments

Viviani’s theorem, although simple in statement, holds a wealth of potential in terms of mathematical investigation as well as proof/justification (see for example Copes & Kahan, 2006). In this article I have focused on the equilateral triangle, but the same basic principle (i.e. the invariance of the sum of the perpendiculars from an arbitrarily positioned interior point) can be extended to any equilateral or equiangular polygon (see for example De Villiers, 2005). In the case of regular polygons, i.e. polygons that are both equilateral and equiangular, the sum of the perpendiculars from an arbitrary interior point is not only invariant, but for a regular $n$-sided polygon the sum will equal $n$ times the apothem (i.e. the perpendicular distance from the centre of the polygon to the midpoint of a side). It is left for the reader to discover why, in the case of a parallelogram (which need not be either equiangular or equilateral), the sum of the perpendiculars to the sides (or extensions of the sides if necessary) from an interior point is also independent of the position of the interior point.

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Teaching Strategies and Activities that Enhance Spatial Visualization

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Introduction
Russell and Nicole are working together on a project in their seventh grade mathematics class. Mrs. Jones, their teacher, asks them to count the number of squares that they can find on a standard 8 by 8 chessboard. Nicole has limited prior experiences with activities such as these, but she easily counts all the small squares and concludes that 64 is the total number of the squares on the 8 by 8 chessboard (or checkerboard). Russell, on the other hand, had played on his Nintendo and Game Boy for five years and was also a self-proclaimed expert at the game of Tetris. In addition to the 64 squares that Nicole found, Russell tells Mrs. Jones that there are additional squares formed by groups of the small squares. Nicole just did not “see” these other squares on her first count. Scenarios like these occur frequently in mathematics classrooms where students are developing the foundation required for higher levels of mathematics. It is important that teachers find ways to enhance students’ spatial sense not only for future success in mathematics but also to enhance students’ interests and talents in maths-related fields such as the physical sciences, architecture, computer-aided design, geographic information systems, and graphic design (Allen, 2003; Lury & Massey, 1999; Mark & Egenhofer, 1994).

How can teachers help students like Nicole “see” hidden or additional objects and geometric shapes using their spatial sense? As mathematics teachers we have observed students in middle school, high school and college mathematics classrooms struggle with a variety of types of mathematics problems requiring the use of spatial visualization. Over time, we have learned that differences in spatial abilities from student to student are due more to prior experiences rather than any lack of inherent ability. Students, regardless of socio-economic status, geographic location, or gender, can increase their spatial abilities when provided with experiences that span multiple levels of abstraction and complexity. Through our work we found that any single activity by itself will do little to improve spatial abilities, but the accumulation of skills gained from numerous, diverse activities can enhance student learning the most. The sequencing of spatial tasks is also important – pre-requisite knowledge and experiences help students move from simple to more challenging spatial tasks.

Many students lack confidence and expect to find learning mathematics a difficult experience. These students are less likely to take risks in class, ask questions, and volunteer to solve a problem on the board, or persist when they make an error. Allowing time for students to discuss their thinking with their peers not only builds confidence, but also encourages an intellectual curiosity to “discover” things and extend their thinking about spatial relations.

In this paper we share an accumulate set of five spatial visualization activities that we have found to aid in improving students’ understanding of geometric concepts, increase their spatial abilities, and build confidence. These activities are carefully sequenced by level of complexity, and we think this is important. Activities 1 and 2 begin with embedded figures and lead, ultimately, into a culmination activity such as Activity 5, finding all the squares on a geoboard. Activities 3 and 4 are transitioning activities because they involve higher-level concepts, thus preparing students for the more complex visualization skills in Activity 5. We based the selection of these specific activities on what students and teachers said were most engaging and effective in helping them “see” solutions. In addition, these activities provide opportunities to build self-confidence by increasing students’ experiences with success, as the tasks become more challenging.

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Activities That Improve Spatial Abilities

We make no claims toward the originality of the activities described in this article, as activities like these have been used in progressive middle school classrooms for many years. However, the process we used in selecting, sequencing and implementing them in the classroom over time is our own original contribution.

**Activity 1: Counting Rectangles**

In this activity, students discover efficient methods for visualizing and counting embedded figures. Students need to accurately count the number of rectangles in a given figure in a reasonable amount of time. Start with a relatively simple shape, such as the rectangles shown in Figure 1. Ask students to “Count all the rectangles of any size” in this figure. Starting with a simple shape allows students to experience success before advancing to more complex drawings. Use co-operative learning strategies such as “Think-Pair-Share” (Kagan, 1989) when giving this problem to students. This technique allows time for students to work on the problem by themselves first (the **think** part) as they count as many individual and grouped rectangles that they can see.

![Figure 1: Counting Rectangles](image.png)

Next, have students pair up with another student, each describing how they counted the rectangles. Students determine which strategy seemed to work best after which they count the rectangles again together (**pair** part) using the most efficient strategy. While students are working in groups, walk around the room asking questions like the following to make students think about their methods of counting and visualizing the rectangles: “How are you positive that you counted every rectangle?” and “How can you be sure that you didn’t count some rectangles more than once?” Finally, have each pair of students demonstrate to the whole class their counting technique as well as their final totals (**share** part). Some students may label each individual rectangle and then systematically list combinations of rectangles that also form a rectangle (logical analytical thinkers tend to choose this method). Other more visual learners may use their hands or pieces of paper to frame the rectangles they are counting by covering all of the others, helping them “see” the combined rectangles better. Others may choose to use different colours to visually show rectangles more clearly. By seeing multiple methods for counting the rectangles, students may discover that there is no one way that works best for everyone. Learning a variety of strategies may enhance their repertoire of problem solving skills. We recommend allowing students to solve problems in small groups in situations such as these where students traditionally struggle to analyze constructions, locate hidden or turned shapes, or devise a strategy to find the solution.

**Activity 2: Counting Trapeziums**

As students demonstrate their spatial skills on the previous tasks, more difficult problems can be created, such as the trapezium activity in Figure 2. This figure may be more challenging to students because they are now working with different angle measurements as well as figures less familiar to them. Ask students to “Find all the trapeziums in the figure.” Tell them that all line segments that appear horizontal are parallel to each other, and that none of the other segments are parallel. The challenge for students is not only to find all the trapeziums but also to find a way of keeping track of the ones they’ve found. Students are working towards efficiency. You may observe some students outlining in color, some shading in figures, and others systematically labeling figures using letters or numbers when counting. Some may begin making an organized list straight away, but others may have no organized system for counting the figures they find. Others may devise a system of counting that looks at particular attributes such as all of the trapeziums in particular “rows” or “columns”, or all of the single trapeziums, then the double trapeziums (formed by only two other trapeziums), and triple trapeziums, and so forth. They may find that the latter...
is the most efficient method of counting trapeziums, and it may be the easiest method to make sure that no trapezium was missed or counted twice; however, we believe that all students should experience the “thrill” of finding the most efficient method for themselves.

Regardless of the method chosen, students are developing their spatial skills in a problem-solving setting. As in Activity 1, we encourage teachers to have students demonstrate their methods for counting and recording the trapeziums so that all students are exposed to multiple strategies.

**Activity 3: Finding the Area of Non-standard Figures**

Activity 3 is best conducted after students have been taught how to find the areas of individual simple polygons (e.g. triangles, rectangles). This activity calls for students to find the area of a trapezium (Figure 3) by working in groups of three or four.

Students are encouraged to visualize the trapezium as a combination of two or more polygons. Again, you may observe students solving this problem in multiple ways. More advanced students may recall the formula for determining the area of a trapezium and immediately apply this knowledge to finding the area. Some students may see the figure as being composed of a rectangle and a triangle (and would calculate these two areas separately and then add them together). Others may divide the whole figure into triangles. Challenge students to find the area using more than one method and then compare their solutions with group members. Students should be able to see that you can use different methods to arrive at the same solution. In a classroom, the trapezium could be drawn on centimetre grid paper to help calculate the area concretely. This can be advantageous for students who are still in need of concrete experiences. By visualizing the trapezium in multiple ways (e.g. rectangle plus a triangle, extended rectangle minus a triangle, multiple triangles), students learn to visualize figures in multiple ways, increasing their spatial abilities.
ACTIVITY 4: PERCEPTUAL CONSTANCY

When the orientation of a shape is changed in a two-dimensional figure, determining if two shapes are actually congruent (same size, same shape) is often difficult when shapes are rotated and/or reflected. For example, in Activity 4, students are asked to determine if the two triangles in Figure 4 are congruent.

![Figure 4: Congruent or Not?](image)

The triangle on the right (triangle ABC) looks like a right-angled triangle to most students and the triangle on the left (triangle EDC) appears to be only an isosceles triangle. Actually, the two triangles are both right-angled isosceles triangles, and they are congruent. Because students most often encounter right-angled triangles in their vertical or horizontal orientation (like triangle ABC in Figure 4), they often expect all right-angled triangles to be oriented that way. Using right-angled triangles (and other polygons) in different orientations during classroom activities helps to develop skills in perceptual constancy. Have students trace triangle ABC on tracing paper and then place it on top of triangle EDC by rotating or reflecting the shapes to see that one triangle “fits” on top of the other. This provides another opportunity for students to use their visual skills concretely. During this activity the teacher’s role is one of facilitator, asking guiding questions to move students along through the activity. As students respond, we have found that reflective listening, where we repeat or rephrase what students tell us, is a more effective tool than just showing them how to move the triangles to see if they are congruent without measuring the sides. The action of sliding, flipping and rotating figures around helps students develop the spatial skills need to “see” solutions.

ACTIVITY 5: SQUARES ON A GEObOARD

Once students have successfully accomplished the first four activities, they are asked to “find all the different size squares on the geoboard” in Activity 5. Students usually do not have too much difficulty finding the four squares that are oriented with their sides parallel to the sides of the geoboard, as shown in Figure 5(a). However, even with encouragement, many students cannot find additional squares with sides that are not parallel to the sides of the geoboard, as shown in Figures 5(b) and 5(c).

![Figure 5: Squares on a Geoboard](image)
Students may need a hint, such as asking them to think of a baseball diamond [Figure 5(b)]. Many will see the diamond by rotating the geoboard. Convincing students that the figure is both a square and a diamond can be difficult until they rotate the geoboard and/or reproduce the figure using tracing paper onto the transformed square. To guide students into viewing the squares on the geoboard from different perspectives, teachers are encouraged to ask questions rather than show students answers. For example, for Figure 5(c), ask: “Have you tried making all the squares from a fixed vertex?” It does little good to just show the solutions in Figures 5(b) and 5(c) to students. Finding the answers themselves is what reinforces their spatial abilities and may provide that “aha” moment for each student when they discover a new way of looking at the problem.

Conclusion

Hands-on activities with mathematics manipulatives and geometric drawings such as the ones described in this article can be effective strategies for helping secondary students improve their abilities to solve a variety of types of mathematics problems requiring the use of spatial visualization. According to a meta-analysis conducted by Sowell (1989), “…mathematics achievement is increased through the long-term use of concrete instructional materials…” (p. 498). The more spatial activities that students experience in earlier grade levels the better their spatial abilities will be when students take future geometry courses and, according to Small and Morton (1983), even in science subjects such as organic chemistry. As educators who have taught middle school through graduate level mathematics students, we believe that spatial experiences should be varied, extensive, exploratory, and increasingly challenging. Furthermore, spatial activities should be used in every classroom at every grade level. Whether a student considers himself or herself visually challenged or not, becoming accustomed to “seeing” things in a spatial manner should be a part of the overall mathematical development of every student. In addition, teachers should consider the developmental appropriateness of each activity, so that students experience success over time. The sequencing of these five activities helps students develop their spatial visualization skills by seeing and manipulating objects and shapes in different positions from different perspectives.

In summary, when teachers provide opportunities for their students through a variety of spatial activities, along with implementation strategies such as co-operative learning, use of manipulatives, and questioning or probing, they provide students with the opportunity to expand their thinking about what they can and cannot do, and provide valuable opportunities to enhance their spatial visualization ability and awareness.

References

-Generalising Cubic Sequences Using a Filtration Method-

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In my experience, most learners have become quite proficient at determining the general expression for a given quadratic sequence. One can build on this proficiency to generalise cubic sequences of the form \( T_n = an^3 + bn^2 + cn + d \) by using a filtration technique. The filtration method provides a means of “filtering out” the cubic part of the general expression for a cubic sequence leaving behind a quadratic “residue”, a general expression for which can be found using any number of conventional approaches.

By way of example, consider the following cubic sequence: 5 ; 17 ; 43 ; 89 ; 161 ; 265... We begin by first establishing that the sequence is indeed cubic by using the standard method of differencing.

\[
\begin{array}{ccccccc}
5 & 17 & 43 & 89 & 161 & 265 \\
12 & 26 & 46 & 72 & 104 & \\
14 & 20 & 26 & 32 & \\
6 & 6 & 6 & \\
\end{array}
\]

The differences are constant at the third level which confirms that the sequence is cubic. Once we’ve determined the third level constant difference we can easily determine the coefficient of \( n^3 \) since the third level constant difference is \( 6a \), where \( a \) is the coefficient of \( n^3 \) in the general cubic expression \( T_n = an^3 + bn^2 + cn + d \). This can easily be established by generalising the differences based on the general expression \( T_n = an^3 + bn^2 + cn + d \) as shown below for the first five terms of the sequence.

\[
\begin{array}{cccccc}
a + b + c + d & 8a + 4b + 2c + d & 27a + 9b + 3c + d & 64a + 16b + 4c + d & 125a + 25b + 5c + d \\
7a + 3b + c & 19a + 5b + c & 37a + 7b + c & 61a + 9b + c \\
12a + 2b & 18a + 2b & 24a + 2b \\
6a & 6a & \\
\end{array}
\]
Thus, for the given cubic sequence \(5 ; 17 ; 43 ; 89 ; 161 ; 265\ldots\) which has a third level constant difference of 6, the coefficient of \(n^3\) is 1. We now know that the general expression of the cubic sequence is \(T_n = n^3 + bn^2 + cn + d\) and can begin the filtration process by subtracting \(n^3\) (i.e. \(an^3\) with \(a = 1\)) from the original cubic sequence. For the first six terms (i.e. \(n = 1\) to \(n = 6\)) the \(n^3\) portion is 1 ; 8 ; 27 ; 64 ; 125 and 216.

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T_n) (i.e. (an^3 + bn^2 + cn + d))</td>
<td>5</td>
<td>17</td>
<td>43</td>
<td>89</td>
<td>161</td>
<td>265</td>
</tr>
<tr>
<td>(T_n - an^3) (with (a = 1))</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>36</td>
<td>49</td>
</tr>
</tbody>
</table>

A general expression for the quadratic residue 4 ; 9 ; 16 ; 25 ; 36 ; 49 can easily be determined by inspection since it is clearly \((n+1)^2\), i.e. \(n^2 + 2n + 1\). Alternatively, the general expression for the quadratic residue can be determined using any number of conventional approaches (see e.g. Samson, 2008). Adding the algebraic expression for the quadratic residue onto the expression for the filtered off cubic portion \((an^3\) with \(a = 1\)) gives a final expression for the original cubic sequence: \(T_n = n^3 + n^2 + 2n + 1\).

Summary of the process:
- Take the given cubic expression and determine the first three levels of differences.
- Determine \(a\), the coefficient of \(n^3\), from the third level constant difference (which is \(6a\)).
- Subtract \(an^3\) from each term of the original cubic sequence.
- Determine a general expression for the quadratic residue in the form \(bn^2 + cn + d\).
- Add \(an^3\) to the expression obtained for the quadratic residue to arrive at a general expression for the original cubic sequence in the form \(T_n = an^3 + bn^2 + cn + d\).

Concluding comments
Given that most learners have become proficient in determining a general expression for a given quadratic sequence, what I hope I have demonstrated in this article is how to build on that proficiency and confidence by extending the basic process to cubic sequences through the use of a filtration method. As a further extension, learners might want to explore how such a filtration method could be used to generalise quartic sequences, i.e. sequences of the form \(T_n = an^4 + bn^3 + cn^2 + dn + e\).

References
Finding problems that students actually enjoy doing can be a challenge. A good place to start might be to consider the problems that previously brought pleasure to you. For me a class of problems that I have enjoyed (and still do) are what Martin Gardiner (Gardiner, 1959, 1961a, 1961b, 1977) called mathematical puzzles, and I have used such puzzles in the classroom with great success. More often than not these kinds of puzzles are used by teachers for their recreational or ‘fun’ element, and it is often difficult to see how such puzzles might relate to the school curriculum. In recent times I have been interested in exploring puzzles that can not only be appropriately incorporated into a lesson, but which form an integral part of the lesson. This often requires a slight modification of the original puzzle in order to render it more suitable, a process which can be far from easy. In this article I present a particular type of puzzle based on Pythagorean triples and, through a process of investigation, show how similar puzzles could be constructed.

THE “LADDER AND BOX” PROBLEM

One often finds that a puzzle only works because of the specially chosen numbers, and once the puzzle has been solved it doesn’t seem to lead anywhere else. If the puzzle is to contribute to mathematical thinking it really needs to be a special case that can be generalised (Mason, 1999; Mason, Burton, & Stacey, 2010). Consider the special case shown alongside (Figure 1) which shows a rectangular box measuring 800 mm by 600 mm resting on a horizontal floor and against a vertical wall. A ladder whose length is 2 metres rests on the box and its two ends just touch the wall and the floor as indicated in the diagram. How far is the foot of the ladder from the wall?

USING AN ARITHMETIC APPROACH

As an initial approach you might choose to explore the possibility that the numbers for this particular puzzle have been specially chosen to produce a “nice” answer. This special case might then be solved using the problem solving strategy of ‘guess and check’. Label the vertices of the triangle formed by the ladder, the wall and the floor, A, B and C respectively, and label the corners of the box P, Q and R. Also convert the lengths to centimetres and construct a line from B to P (Figure 2). Since $\angle ABC = 90^\circ$, $AC$ is the diameter of the circumcircle through $A$, $B$ and $C$. $\triangle APB$ is a 3, 4, 5 right-angled triangle and $PB$ is thus 100 cm. Point P is therefore the centre of the circumcircle and we have $PA = PB = PC = 100$. There are four small congruent triangles, $\triangle AQP$, $\triangle BOP$, $\triangle PRB$ and $\triangle PRC$, from which it follows that $AQ = 60$ cm and $RC = 80$ cm. The solution to the problem (How far is the foot of the ladder from the wall?) can now clearly be seen from the diagram. $BC = BR + RC = 80 + 80 = 160$ cm.
We can therefore deduce that the distance of the foot of the ladder from the wall is $BC = 2a$.

**Using an Algebraic Approach**

I use the term ‘arithmetic approach’ to mean that the solution is developed without the use of algebra. When algebra is introduced into the solution I refer to it as an ‘algebraic approach’. Let’s re-label our diagram using the letters $a$, $b$ and $c$ for the lengths $QP$, $PR$ and $AC$ respectively (Figure 3). The algebraic solution is produced by ensuring $P$ is at the centre of the circumcircle and therefore $PA = PB = PC$. In this case the four triangles are congruent and $\frac{1}{2}c = \sqrt{a^2 + b^2}$. Also $AQ = b$ and $RC = a$. We can therefore deduce that the distance of the foot of the ladder from the wall is $BC = 2a$.

**Variation on a Theme**

By ensuring that $P$ is at the centre of the circumcircle the special case can be used to generate a similar problem and this will produce a “nice” answer if a Pythagorean triple is the starting point. You often find that a Pythagorean triple has been used for the basis of a puzzle of this nature and that an arithmetic approach will lead to the solution. However, if the above restriction is lifted and $P$ is no longer the centre of the circumcircle then the four triangles will no longer be congruent. By way of example consider Figure 4 which shows a ladder with length $c$ and the box with cross-sectional dimensions $a$ and $b$. The unknown lengths, $RC$ and $AQ$, are $x$ and $y$ respectively.

Since the triangles $ABC$, $AQP$ and $PRC$ are similar we have: $\frac{y}{a} = \frac{b}{x}$ and thus $xy = ab$ \hspace{1cm} (1)

In addition, Pythagoras’ theorem gives: $(x + a)^2 + (y + b)^2 = c^2$ \hspace{1cm} (2)

Equations (1) and (2) can be simplified further if we let $a = b$, in other words if the box has a square cross-section, but before we carry out this simplification let us consider the following puzzle which represents a special case when $P$ is no longer the centre of the circumcircle through $A$, $B$ and $C$.

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A SPECIAL CASE

The diagram (Figure 5) shows a box with a square cross-section. The box has dimensions of 48 cm by 48 cm and rests on a horizontal floor against a vertical wall. A ladder whose length is 140 cm rests on the box and its two ends just touch the wall and the floor. How far is the foot of the ladder from the wall?

![Figure 5](image)

You would expect the numbers for this puzzle to have been specially chosen to produce a “nice” answer. This special case might also then be solved using the problem solving strategy of ‘guess and check’. Firstly you recognise that the three right-angled triangles $ABC$, $AQP$ and $PRC$ are similar triangles. If you had some past experience with puzzles with right-angled triangles you might reason that the three triangles might be 3, 4, 5 triangles. Past experience might tell you that this is a favourite for puzzle writers. If $\Delta AQP$ is a 3, 4, 5 triangle then $AQ = 36$ (i.e. $3 \times 12$) and $AP = 60$ (i.e. $5 \times 12$). If $\Delta PRC$ is a 3, 4, 5 triangle then $RC = 64$ (i.e. $4 \times 16$) and $PC = 80$ (i.e. $5 \times 16$). Now $AC = AP + PC = 60 + 80 = 140$ and we know this is indeed the case. The solution is therefore $BC = 48 + 64 = 112$ cm. So the foot of the ladder is 112 cm from the wall.

A GENERAL SOLUTION

This special case raises questions. You might wonder, for instance, how hard it would be to invent other special cases. Perhaps it is only the 3, 4, 5 triangle that will produce a special case. Let us explore this by way of developing an algebraic solution. ‘Specialising and generalising’ is a process promoted by John Mason (Mason, 1999; Mason et al., 2010) and this process is particularly appropriate here. He uses four key words, $\text{STUCK, AHA, CHECK}$ and $\text{REFLECT}$ as a framework to proceed. Let us try to develop a general solution and then test our special case (Figure 5) against the general solution.

Figure 6 shows a general representation of Figure 5. The ladder has length $c$ and the box, with a square cross-section, has dimensions $a \times a$. $RC$ and $AQ$, are $x$ and $y$ respectively.

![Figure 6](image)
Some simple geometry proves that Δs ABC, AQP and PRC are similar right-angled triangles. From triangles AQP and PRC we have:

\[ \frac{AQ}{QP} = \frac{PR}{RC} \]

Therefore, \( \frac{y}{a} = \frac{x}{x} \), and hence \( xy = a^2 \).

Applying Pythagoras’ theorem to: ΔABC:

\[
(x+a)^2 + (y+a)^2 = c^2
\]

\[
x^2 + 2ax + a^2 + y^2 + 2ay + a^2 = c^2
\]

\[
x^2 + y^2 + 2ax + 2ay + 2a^2 = c^2. \quad \text{But as } xy = a^2 \text{ then } 2xy = 2a^2
\]

\[
x^2 + y^2 + 2a(x + y) + 2xy = c^2
\]

\[
x^2 + 2xy + y^2 + 2a(x + y) = c^2
\]

\[
(x + y)^2 + 2a(x + y) = c^2. \quad \text{Complete the square on the left hand side}
\]

\[
(x + y)^2 + 2a(x + y) + a^2 = c^2 + a^2
\]

\[
(x + y + a)^2 = a^2 + c^2
\]

\[
x + y + a = \sqrt{a^2 + c^2}
\]

Since this is a practical problem about a box and a ladder, the lengths \( x \), \( y \) and \( a \) are all positive and we can ignore the negative square root. Our two simultaneous equations are thus:

\[
xy = a^2 \quad \text{and} \quad x + y + a = \sqrt{a^2 + c^2}
\]

These equations can be combined to produce a quadratic equation:

\[
y = \sqrt{a^2 + c^2} - a - x
\]

\[
x\left(\sqrt{a^2 + c^2} - a - x\right) = a^2
\]

When this equation is rearranged it becomes:

\[
x^2 + \left(a - \sqrt{a^2 + c^2}\right)x + a^2 = 0 \quad (3)
\]

This quadratic equation should give the solution to our special case if we have made no mistakes, and will be a step in our CHECK process. As we tackled this problem we probably had several moments when we were STUCK, and these would have been followed with AHA moments as we discovered other ways forward.

We can now use the ladder length and the box dimensions of the special case given earlier in Figure 5 and substitute them into equation (3). With \( c = 140 \) and \( a = 48 \) equation (3) becomes:

\[
x^2 + \left(48 - \sqrt{48^2 + 140^2}\right)x + 48^2 = 0
\]

It’s certainly encouraging (an AHA moment) that the square root simplifies nicely: \( \sqrt{48^2 + 140^2} = 148 \)

We thus have \( x^2 + (48 - 148)x + 2304 = 0 \) which simplifies to:

\[
x^2 - 100x + 2304 = 0
\]

This quadratic equation factorises rather nicely (another AHA moment):

\[
(x - 64)(x - 36) = 0
\]

\[
\therefore \quad x = 64 \quad \text{or} \quad x = 36
\]

And there it is – the same solution emerges. Of course the advantage of the algebraic approach is that it produces both answers. When an arithmetic approach is used the tendency is often to stop after a single solution has been found, happy that the numbers chosen satisfy the conditions.

DEVELOPING FURTHER SPECIAL CASES

The algebraic solution certainly produces the special case when the numbers are substituted into equation (3). However, I would like to take this a step further and investigate how we might generate whole number solutions to produce additional similar puzzles.

Let us begin by putting together what we have found thus far to construct a puzzle about a box with a square cross-section and ladder. We found that for our puzzle the numbers came from the 3, 4, 5 right-angled triangle. We conjecture that Pythagorean triples will generate whole number puzzles. How would one go about creating the puzzle we have been using for our investigation? With reference to Figure 6, since triangles $AQP$ and $PRC$ are similar we have:

$$\frac{y}{a} = \frac{a}{x}$$

In order for $a$ to be the length of non-corresponding sides in the two similar triangles the simplest value for $a$ is $3 \times 4$. From this we can deduce that $\triangle AQP$ will have $AQ = 9$ and $AP = 15$ and $\triangle PRC$ will have $RC = 16$ and $PC = 20$. This can be seen in Figure 7.

Our puzzle (Figure 5) is thus created by scaling Figure 7 up by a factor of 4.

Let us now specialise with the 5, 12, 13 Pythagorean triple. In this case $a = 5 \times 12 = 60$ and the corresponding diagram is shown in Figure 8.
From this result we can create the following similar puzzle:

The diagram (Figure 9) shows a box with a square cross-section. The box has dimensions of 60 cm by 60 cm and rests on a horizontal floor against a vertical wall. A ladder whose length is 221 cm rests on the box and its two ends just touch the wall and the floor. How far is the foot of the ladder from the wall?

As previously, we have $xy = a^2$ and $x + y + a = \sqrt{a^2 + c^2}$ which can be combined to produce quadratic equation (3):

$$x^2 + \left(a - \sqrt{a^2 + c^2}\right)x + a^2 = 0$$

Substituting $a = 60$ and $c = 221$ yields:

$$x^2 - 169x + 3600 = 0$$

$$(x - 144)(x - 25) = 0$$

∴ $x = 144$ or $x = 25$

The foot of the ladder is therefore either 144 + 60 = 204 cm or 25 + 60 = 85 cm from the wall.

Our conjecture can now be stated more clearly. We start off with a Pythagorean triple and choose the lengths of the box as the product of the two shorter sides of the right-angled triangle. The length of the ladder can be deduced and its length will be a whole number. If the length of the ladder is not sensible a suitably chosen scale factor will provide a length that is reasonable. It so happened that the length of 221 cm (2.21 metres) for the length of the ladder is quite reasonable, so no scale factor is necessary. However, with the 3, 4, 5 right-angled triangle a scale factor of 4 allowed us to come up with a puzzle with a ladder of length 140 cm (1.40 metres) which seemed reasonable at the time. Using this strategy the quadratic equation obtained will factorise and lead to two whole number solutions for the distance of the foot of the ladder from the wall.

We have shown this conjecture is valid for our two simplest Pythagorean triples, but will it be valid for all Pythagorean triples? I will leave readers to verify this if they wish to.

**CONCLUDING COMMENTS**

Our conjecture can be viewed from an algebraic perspective as shown in Figure 10. Let $a, b$ and $\sqrt{a^2 + b^2}$ be the sides of a right-angled triangle, then the lengths of the sides of the box will be $ab$ and the length of the ladder will be $(a + b)\sqrt{a^2 + b^2}$. The two possible answers for the distance of the foot of the ladder from the wall will be either $a(a + b)$ or $b(a + b)$. If $a$ and $b$ are the two smaller numbers of a Pythagorean triple, the length of the ladder will be a rational number.

**REFERENCES**


A Study of a Multiple-Strategy Approach to Locating the Centre of a Circle

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Problem-solving plays a crucial role in mathematics education. One of the aims of teaching through problem-solving is to encourage students to refine and build their own processes over time as their experience allows them to discard some ideas and makes them aware of other possibilities (Carpenter, 1989). In addition to developing their knowledge, students also acquire an understanding of when it is appropriate to use particular strategies. Emphasis is placed on making students responsible for their own learning rather than letting them feel that the methods that they are using are the inventions of others. Considerable importance is placed on exploratory activities, observation and discovery, and trial and error. Students must develop their own ideas, test them, discard them if they are not consistent, and try something else (National Council of Teachers of Mathematics [NCTM], 1989). Students become more involved in problem-solving by formulating and solving their own problems, or by rewriting problems in their own words to aid their understanding. Crucially, students are encouraged to discuss the processes that they are using to improve their understanding, gain new insights into the problem, and communicate their ideas (Thompson, 1985; Stacey & Groves, 1985).

Polya (1990) proposed four steps in mathematical problem solving: (a) understanding the problem, (b) making a plan, (c) carrying out the plan, and (d) looking back. Some examples of applying the Polya's model can be found in Poon & Wong (2011) and Poon (2011). Although these steps focus on solving problems, Polya's objective was to teach students to think by mean of emphasizing reflection and looking back in the fourth step. “By reconsidering and re-examining the result and the path that led to it, students could consolidate their knowledge and develop their ability to solve problems” (Polya, 1990, p. 15).

Open-ended problems have been regarded as powerful tools for teaching problem solving in mathematics (see for example Craft, 2000), especially in the “looking back” part in the problem solving process. Open-ended problems allow students to think about different strategies to tackle the problems, and are a useful way of consolidating students’ knowledge. Open-ended mathematics problems are also important in terms of developing students’ problem solving skills (see for example Monaghan et al., 2009). The usefulness of encouraging an open-ended approach in helping and assessing students’ higher order thinking in mathematics has also been examined (Becker & Selter, 1996; Becker & Shimada, 1997; Shimada, 1977). Open-ended problems are characterised as problems that are formulated so that there is more than one solution (Becker & Shimada, 1997) or that lend themselves to more than one solution path (Cifarelli & Cai, 2005). These types of problems provide students with challenging experiences for finding new, alternative and creative solutions (Ho & Hedberg, 2005).

An interesting open-ended mathematics problem is that of locating the centre of a circle. Although the concepts of a circle and its centre are familiar to most people, the simple question of how to find the centre of a circle by geometrical construction is hard for many students. This paper focuses on the open-ended problem of locating the centre of a circle by means of geometric construction, and discusses a variety of strategies to tackle the problem. This open-ended problem may also serve as a problem solving activity for senior high school students.
The Problem

Suppose that we are given a circle \( c \), but we do not know where the centre \( O \) is. How can we find the centre \( O \)? It is clear that the circle has exactly one centre, and so the solution to this problem is unique. Nevertheless, there are many different ways to find the centre of the circle.

Geometric Construction

Let us first follow the famous mathematician, Euclid, confining ourselves to the use of straight edge and compasses only. We can use such dynamic geometry software (DGS) as The Geometer’s Sketchpad or GeoGebra to aid our construction. The advantage of using DGS instead of traditional straight edge and compasses is that the drawings are not fixed. We can easily modify the positions of the points and lines we have drawn in the software. Moreover, the numerical values such as the length of a line segment and the size of an angle can be easily measured. When we move the points or lines in the software, these values will change accordingly. Such functions allow us to carry out various investigations, which may also facilitate our insights in problem solving. In the following geometric constructions, all the pictures are drawn by using GeoGebra (which can be freely downloaded from www.Geogebra.com). For manual construction with pencil and paper, the various methods outlined in this article assume knowledge of a number of basic construction techniques, specifically (i) constructing a perpendicular bisector, (ii) copying an angle, (iii) drawing a line parallel to another line, and (iv) bisecting an angle. These are rudimentary techniques that can be found in most standard geometry texts.

Method 1 (Two chords method)

Since the perpendicular bisector of a segment is the path of all points equidistant from the endpoints of the segment, it follows that \( O \), the centre of the circle, which by definition is equidistant from the endpoints of any chord, must necessarily lie on the perpendicular bisector of the chord. We thus have the result that the perpendicular bisector of a chord passes through the centre of the circle. This is our starting point for the “two chords method”.

In the geometric construction, we start with a chord \( XY \) of the circle \( c \), and construct the perpendicular bisector \( l \) of \( XY \). This line \( l \) thus passes though the centre \( O \). We repeat the same process for another chord \( X'Y' \) and obtain its perpendicular bisector \( l' \). Since \( O \) lies on both of the lines \( l \) and \( l' \), the point of intersection of \( l \) and \( l' \) must be \( O \).

For simplicity, we may, in particular, also consider the points of intersection of \( l \) and \( l' \) as our \( X' \) and \( Y' \). In such a case the chord \( X'Y' \) is in fact a diameter of \( c \) as shown in the figure alongside.

From the above method it should be clear that the problem of finding the centre of a circle will be solved as soon as we find a diameter of the circle \( c \), since the centre of the circle will be the point of intersection of two such diameters. From now on we shall thus simply focus on constructing a line that passes through the centre \( O \), since the construction of two such lines will allow us to pinpoint the centre of the circle. Alternatively, once we have constructed one line that passes through the centre of the circle we can simply bisect the diameter formed in order to pinpoint the centre of the circle.
Method 2 (Angle bisector of two equal chords)

Suppose that $XY$ and $X'Y'$ are two intersecting chords such that $XY = X'Y'$. It can be verified that $O$ will lie on the angle bisector $l$ of $\angle YPY'$, where $P$ is the point of intersection of $XY$ and $X'Y'$. Let $M$ and $M'$ be the feet of the perpendiculars dropped from $O$ to $XY$ and $X'Y'$ respectively:

- $OM = OM'$, equal chords, equidistant from centre
- $\angle OMP = 90^\circ$, given
- $\angle OMP' = 90^\circ$, given
- $OP = OP$, common side
- $\triangle OMP \equiv \triangle OMP'$, RHS
- $\angle OPM = \angle OPM'$, corr. $\angle s$, $\equiv \angle s$

Therefore, $OP$ is indeed an angle bisector of $\angle YPY'$.

In the geometric construction, we start with a chord $XY$ of $c$. We then choose another point $X'$ on $c$. Using $XY$ as the radius, draw a circle centred at $X'$ which intersects $c$ at some point $Y'$, and let $P$ be the point of intersection of $XY$ and $X'Y'$. Finally construct the angle bisector $l$ of $\angle YPY'$, which must pass through the centre of the circle $c$.

For the sake of simplicity, $X$ and $X'$ can even be chosen to be the same point. In this case the three points $X$, $X'$ and $P$ coincide. The centre $O$ will lie on the angle bisector $l$ of $\angle YXY'$.

Method 3 (Complementary angles method)

Given a chord $XY$ and a point $P$ on $c$, the angle $\angle XPY$ is known as the angle at the circumference subtended by $XY$. If $O$ is the centre of $c$, then the angle $\angle XOY$ is called the angle at centre. It is well known that the angle at centre is twice the angle at circumference. That is, $\angle XOY = 2\angle XPY$ (\angle at centre twice \angle at circumference).

Concerning our original problem, we observe that:

- $OX = OY$, radii
- $\angle OXY = \angle OYX$, base $\angle s$, isos. $\Delta$
- $\angle XOY = 2\angle XPY$, $\angle$ at center twice $\angle$ at circ.
- $\angle XOY + \angle OYX = 180^\circ - \angle XOY$
- $\angle XOY + \angle OXY = 180^\circ - 2\angle XPY$
- $2\angle XOY = 180^\circ - 2\angle XPY$
- $\angle OXY = 90^\circ - \angle XPY$

Hence, the angles $\angle OXY$ and $\angle XPY$ are complementary, in the sense that the sum of these two angles equals a right angle.

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1 This lemma is left for the reader to verify. It follows from the congruency of triangles formed by the equal chords and radii from the centre of the circle (SSS), from which the corresponding perpendicular heights from $O$ must also be equal.

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Thinking backwards, we know that a straight line $l$ through $Y$ must contain the centre $O$ if the angle $\beta$ made by $l$ and $XY$ is complementary with the angle at the circumference, $\angle XPY$.

Consider the angle at the circumference $\angle XPY$ subtended by the chord $XY$. Construct the perpendicular line to $PX$ through the point $P$, making an angle $\alpha = \angle YPA$ with the chord $PY$. This can be done by extending the line $PX$, regarding it as a straight angle at $P$, and drawing its angle bisector. We then copy the angle $\alpha$ to the chord $XY$ at the point $Y$, such that $\beta = \angle CYB = \alpha$. It is evident that the line $l$ joining $C$ and $Y$ must pass through $O$, because the angle $\beta$ and $\angle XPY$ are complementary.

We can then obtain the centre $O$ by repeating the process with respect to point $X$.

In the foregoing geometric construction, we consider an acute angle at circumference, $\angle XPY$. The same process can be applied to an obtuse angle, but in such a case we must regard $\alpha = \beta$ as negative angles.

For the sake of simplicity, we may even start with a right angle at the circumference, $\angle XPY = 90^\circ$. Then the chord $XY$ will become a diameter of the circle $c$ since the angles $\alpha = \beta$ vanish and $l$ coincides with the line $XY$. We then construct the midpoint $O$ of the diameter $XY$.

In fact, the angle at circumference $\angle XPY$ is a right angle when and only when the chord $XY$ is a diameter. This result is generally known as $\angle \text{in a semicircle}$.

**Method 4 (Tangent method)**

This method makes use of the tangent of the circle. It is well known that a straight line is a tangent of a circle if and only if it touches the circle at exactly one point. Let $X$ be a point on the circle $c$. By using the convexity of the circle, we can show that the tangent line at $X$ must lie entirely outside $c$, except for the point of contact $X$. The radius $OX$ is, intuitively, perpendicular to the tangent at $X$. This is indeed true, because if $P$ is another point on the tangent line, then we always have $OP > OX$. The inequality degenerates into an equality when and only when $P$ coincides with $X$. Now that $OX$ assumes the shortest distance from $O$ to the tangent, it must also be perpendicular to it.

It should now be evident that the perpendicular line $l$ to the tangent at the point $X$ must pass through the centre $O$. The line $l$ is known as the normal of $c$ at the point $X$. In the geometric construction we construct the tangent line at a point $X$ on the circle $c$. Through $X$ we then draw the normal line $l$ perpendicular to this tangent. It can easily be done by considering the tangent line as a straight angle at $X$ and drawing its angle bisector. The centre $O$ is determined as soon as we construct two such normal lines.
Method 5 (Angle bisector of two tangents method)

Similar to the foregoing method, this method also make use of the tangent properties of the circle. Let \( P \) be a point outside the circle \( c \). Through \( P \) draw two tangent lines \( PX \) and \( PY \), \( X \) and \( Y \) being the respective points of contact. Now, since \( O \), the centre of the circle, is by definition equidistant from both \( X \) and \( Y \), it must necessarily lie on the angle bisector of the two tangents since the angle bisector is the path of all equidistant points from the two tangents.

Discussion on the construction of tangent lines

Although software such as GeoGebra allows one to draw a tangent to a circle directly from the menu, Method 4 and Method 5 assume paper and pencil constructions and depend heavily on the construction of tangents. This can readily be accomplished if we already know the position of the centre \( O \).

Let \( P \) be a point outside \( c \). We join the line segment \( OP \) and find its midpoint \( M \). Considering \( M \) as the centre, we draw a circle with radius \( OM \), intersecting the given circle \( c \) at the two points \( X \) and \( Y \).

\[
\angle OXP = 90^\circ \quad \angle \text{in semicircle}
\]

\[ PX \text{ is tangent to } c \text{ at } X \quad \text{tangent } \perp \text{ radius} \]

Similarly, we see that \( PY \) is a tangent line of \( c \) at the point \( Y \).

However, this particular construction of the tangent line assumes we already know the position of the centre \( O \). Is it possible to draw the tangent without using the centre \( O \)? Let us refine our strategy. Given a point \( P \) outside the circle \( c \) (see diagram below), we first draw an arbitrary line through \( P \), intersecting \( c \). If this line happens to intersect \( c \) at exactly one point, then this line is a tangent, and we are done. So, we can assume that the line passes through two distinct points \( X \) and \( Y \) on \( c \). In such a case, the straight line \( PXY \) is a secant line of \( c \).

Now draw another line through the point \( P \), intersecting \( c \). Once again, if this line happens to intersects \( c \) at exactly one point, then this line is a tangent, otherwise we can assume that the line passes through two distinct points \( X' \) and \( Y' \) on \( c \).

Considering the two triangles \( \triangle PXY' \) and \( \triangle PX'Y \), we have:

\[
\angle PYX' = \angle PY'X \quad \text{as in same segment}
\]

\[
\angle XPY = \angle XPY' \quad \text{common } \angle
\]

\[
\Delta PXY' \parallel \Delta PX'Y \quad AA
\]

\[
\frac{PX}{PX'} = \frac{PY'}{PY} \quad \text{corr. sides, } \parallel \Delta
\]

\[
(PX')(PY) = (PX)(PY')
\]

What is the geometric meaning of this equality? We have, in fact, shown that \((PX)(PY)\) is a constant, independent of the choice of the secant line. The product \((PX)(PY)\) is known as the power of the circle \( c \) at the point \( P \) and the above equality is known as the circle power theorem.

As our aim is to construct a tangent line through \( P \), we want the two points of intersection \( X' \) and \( Y' \) to coincide. In such a case the power of circle \( c \) at point \( P \) satisfies \((PX')(PY') = (PX')(PY') = (PX)(PY)\). We are happy if we can obtain the geometric mean \( \sqrt{(PX)(PY')} = PX' \) through geometric construction. This can be accomplished by using similar triangles.

We start with the secant line \( PXY \). Treating the midpoint \( M \) of the line segment \( PY \) as the centre, we draw a circle with radius \( MP \). Through the point \( X \) construct a perpendicular line to \( PY \), cutting this circle at the
point \( Q \). Then, construct the circle centred at \( P \) with radius \( PQ \), intersecting the given circle \( c \) at the point \( X' \). Then:

\[
\begin{align*}
\angle PXQ &= 90^\circ \quad \text{by construction} \\
\angle PXY &= 90^\circ & \angle \text{in semicircle} \\
\angle XPQ &= \angle QPY \quad \text{common} \\
\Delta PXQ &\parallel \Delta PXY & \text{AA} \\
PX &= \frac{PQ}{PY} & \text{corr. sides} \parallel \Delta s \\
\sqrt{(PX)(PY)} &= PQ \\
&= PX' \quad \text{radii}
\end{align*}
\]

It is evident that the line \( PX' \) is our sought after tangent.

**Conclusion**

The study of circle properties is quite often regarded as an interesting yet difficult topic in Euclidean geometry. We hope that the open-ended problem presented here may facilitate the teaching and learning of this topic and encourage deeper understanding of the properties of the circle. Though the statement of this problem appears to be very simple, it is astonishing to see how many different possible solutions there are. The process of finding multiple solutions to a problem is important in the learning of geometry as it helps students to see the relationships between different concepts.

We also encourage the use of ICT in problem solving. In this open-ended problem, the aid of DGS can greatly ease the process of geometric construction. Not only can the drawings be made quickly and accurately, but many of the auxiliary elements of the construction process are hidden from view, thus focusing attention on the critical conceptual issues. We believe that this treatment of the problem may allow students and teachers to appreciate the beauty of mathematics, and to enjoy the challenges in problem solving.

**References**


Considering Terminology and Notation of the Decimal System

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Have you or any of your students noticed that terms on the left and right side of the decimal point can appear to be inconsistent? For example, the hundreds’ digit is 3 places to the left of the decimal point, while the hundredths’ digit is 2 places to the right of the decimal point. Students can sometimes find the terminology confusing since tens are not paired with tenths, hundreds are not paired with hundredths, and thousands are not paired with thousandths. This reflects an innate expectation for symmetric terminology.

Consider the following example (for illustration we include commas for the grouping of three digits on both the left and right of the decimal):

\[ 9,876.123,455,55 \]

We have the following pairs:

<table>
<thead>
<tr>
<th>1\textsuperscript{st} place to the left of the decimal point: Units’ place (here 6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1\textsuperscript{st} place to the right of the decimal point: Tenths’ place (here 1)</td>
</tr>
<tr>
<td>2\textsuperscript{nd} place to the left of the decimal point: Tens’ place (here 7)</td>
</tr>
<tr>
<td>2\textsuperscript{nd} place to the right of the decimal point: Hundredths’ place (here 2)</td>
</tr>
</tbody>
</table>

For anyone already accustomed to the decimal system, it requires little if any thought. The system is based on multiples of the units’ place, where each place is ten times the place to the right. The tens’ place is ten times the units’ place, and the units’ place is ten times the tenths’ place, and so on.

But let us imagine what a decimal system would look like that is consistent in terminology and notation. We could represent the decimal system number 12,345.6789 as:

\[ 12,345.6789 \]

Here the symbol under the 5 is a “pivot” used to represent and identify the units’ place. Notice that the word “pivot” is used to represent the symmetry of notation that is desirable on either side of the units’ place. With this arrangement, we now have the following pairs:

<table>
<thead>
<tr>
<th>1\textsuperscript{st} place to the left of the pivot: Tens’ place (here 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1\textsuperscript{st} place to the right of the pivot: Tenths’ place (here 6)</td>
</tr>
<tr>
<td>2\textsuperscript{nd} place to the left of the pivot: Hundreds’ place (here 3)</td>
</tr>
<tr>
<td>2\textsuperscript{nd} place to the right of the pivot: Hundredths’ place (here 7)</td>
</tr>
</tbody>
</table>

This notation allows us to use commas, normally only shown to the left of the decimal point, on \textit{either} side of the “pivot” consistently. To the right of the units’ place, "reversed" commas are used to match the usual commas on the left side of the pivot.

So we have a system that is symmetric for terminology (place names) and notation (commas) around the units’ place pivot. Admittedly, this hypothetical system takes a little getting used to if you are already perfectly familiar with the standard decimal notation. However, upon reflecting on the notational and visual symmetry of this system, teachers may have a better understanding of why students struggle so much with our traditional system.
Counting Lattice Points

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The following problems challenge students to organize their work, recognize patterns, make generalizations, and apply a variety of algebraic concepts and formulas, all within the context of lattice points on a grid. This activity was inspired by the classic “painted cube” problem in which a painted cube is cut into smaller identical cubes and the problem is to determine the number of cubes with 0, 1, 2 and 3 faces painted. When students have completed the Counting Lattice Points activities, they should be encouraged to investigate the “painted cube” problem.

The activities in this article are designed to lead students to generalize about the lattice points in a regular array of such points. Let us begin with a square array of lattice points as shown in Figure 1. For clarity we will refer to the black dots as lattice points while the line segments linking lattice points will be referred to as segments. In addition, the dimensions of the grid refer to the unit lengths of the sides. Figure 1 thus shows a $3 \times 3$ grid.

Notice that some lattice points are directly connected to two segments, some are connected to three segments, and yet others are connected to four segments. The first part of the activity encourages students to explore the lattice shown in Figure 1:

1. How many lattice points does the $3 \times 3$ grid contain?
2. How many lattice points are directly connected to (a) two segments, (b) three segments, (c) four segments?
3. Show that the sum of your answers to (2) is the same as your answer to (1).

Encourage students to repeat the investigation for other sized square arrays such as $4 \times 4$, $5 \times 5$ or $6 \times 6$ grids. Dot paper is useful for this exercise. As they investigate and discuss their findings, encourage students to pay particular attention to the positions of those lattice points that have two, three or four segments. Once you feel your students have a good sense of the structural arrangement of the grid, move them on to the following generalisation activity for an $n \times n$ grid:

1. Write down an algebraic expression for the number of lattice points in an $n \times n$ grid.
2. Write algebraic expressions for how many lattice points are directly connected to (a) two segments, (b) three segments, (c) four segments.
3. Show that the sum of your answers to (2) is algebraically equivalent to your answer in (1).
Students should be able to reason that in an $n \times n$ grid:

- $4$ lattice points will have two segments (the lattice points at each of the four corners).
- $4(n - 1)$ lattice points will have three segments (the lattice points along the outside edge of the grid between the corner lattice points).
- $(n - 1)^2$ lattice points will have four segments (the inside lattice points, i.e. those not along the outside edge).

Once students have explored a square array of lattice points, extend the idea by getting them to repeat the investigation with a rectangular grid such as the $4 \times 5$ grid shown in Figure 2.

![Figure 2: A rectangular array of lattice points.](image)

Once again, students should be encouraged to explore a variety of rectangular arrays of different sizes before generalizing for an $m \times n$ grid.

As a final extension, students could also consider triangular or hexagonal arrays of lattice points. Sheets of isomorphic dot paper would be useful for this investigation.

![Figure 3: Triangular and hexagonal arrays of lattice points.](image)
Problems with Word Problems in Mathematics

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INTRODUCTION

Many teachers complain that learners find word problems in mathematics more difficult than straight computation, and that many learners dislike and even fear word problems. Different reasons are given for this phenomenon, of which the most common is that learners cannot read with understanding and therefore do not know what is required of them. One way to resolve this dilemma is simply to sidestep it – teach the methods, tools, techniques, and formulas which society believes make up mathematical knowledge and then test and examine in such a way that this kind of knowledge will ensure a good pass rate. Although this was a popular and common choice in the past, extensive research and practical (and personal) experience show clearly that this type of mathematics education is no real education at all and does not equip learners to deal with mathematical problems or to cope with further education in mathematics. For learners of all ages to become successful users of mathematical tools and mathematical thinking and problem solving strategies, they need to learn their mathematics with understanding, be able to relate to it, and try to make personal and collective sense of what they are learning. This can only happen when the mathematical ideas which we want to develop are embedded in situations that provide the possibilities of making connections to previous experiences, knowledge, and needs. For this, we as educators need word problems, i.e. situations which can give reasons for and meaning to the mathematics we want learners to acquire. However, although we therefore need word problems, the reasons why learners find word problems difficult are extremely complex. One of these reasons may be called the “straightforward” language problem – the reader’s lack of knowledge of the words used or the syntax where second-language or third-language speakers are obviously at a great disadvantage. However, in this paper I explore some of the other reasons why word problems are found to be difficult and unpleasant. I believe that these other reasons also have deep ties with literacy, because they involve perceptions and beliefs held by the learner about mathematics, about what is expected of her, and about her teacher, as well as the teacher’s beliefs about mathematics, how mathematics is learnt, and about appropriate learner behaviour.

WORD PROBLEMS – A COMPLEX ISSUE

It sounds reasonable to assume that, when learning mathematics, the learner’s ability to understand the language of instruction and also her level of reading comprehension play an important part in successful learning. We can have no quarrel with this, but there is a danger that functional literacy (including reading comprehension) can be interpreted too superficially, without taking into account all the many factors that may prevent the learner from making sense of what she is reading. Take for example the findings of the Third International Mathematics and Science Study–Repeat (TIMSS–R) (Howie, 2001) which show that, in South Africa, learners who seldom or never spoke the language of the test at home achieved lower scores than those who did. Yet in other countries, for example Malaysia and Morocco, the opposite was true – an observation that runs contrary to common sense expectations.

¹ Hanlie Murray passed away in January 2012. This previously unpublished article (prepared for a symposium at the 2001 International Literacy Conference in Cape Town) is published here posthumously as a tribute to her memory.
To highlight and illustrate how very complex a seemingly straightforward task of reading a piece of text really is, I would like to use an example from primary school mathematics. Let me state immediately that we shall not be looking at text which describes a mathematical concept or process, but at simple descriptions of “real life” situations as posed to young children in the form of word problems for them to solve. No mathematical terminology except number names and number symbols are used. Such contextualised problems, sometimes called story sums, should form a major part of the young child’s mathematical learning programme. When children are encouraged to solve word problems, their informal knowledge is elicited, they can relate to and make sense of their school mathematics, and they gain experience with a variety of mathematical structures and processes that can be developed and refined (Murray, Olivier & Human, 1998). Such word problems should therefore necessarily be simple and involve everyday situations and objects that young children can identify with. Therefore, when a teacher poses a word problem, either verbally or in writing, and the learners do not respond, she may say that they do not (or cannot) listen or that they do not (or cannot) read or do not read with sufficient care. Is this really the reason?

**WORD PROBLEMS – BARRIERS TO UNDERSTANDING**

To ensure a common understanding, the examples of problems will be limited to early mathematics. For the purposes of this paper, we shall take a word problem as a problem with a story or (pseudo) real life situation as context, and not simply a calculation. For a child to understand and respond to the problem posed, the language and grammatical constructions used when the word problem is formulated are obviously crucially important. This is self-evident. In addition, the following factors have been proved to be as important (compare Fischbein, Deri, Nello & Marino, 1985, p. 5).

1. The mathematical structure of the problem
2. The number sizes and kinds of numbers involved
3. The context used for the problem, for example, shopping, sports, a trip
4. The learner’s beliefs about what mathematics is and what the teacher expects from her
5. The teacher’s beliefs about what mathematics is, how mathematics is learnt, and how children learn mathematics.

The above factors can be divided into two very broad groups. The first group involves subject-related factors and the other group the socio-psychological factors. However, we need to keep in mind that the two groups cannot really be separated, as learners’ and teachers’ beliefs and attitudes influence not only the quality of the mathematics which is learnt, but also the kind of mathematics which is learnt.

1. **The mathematical structure of the problem**

There is an obvious difference between the mathematical structures of the following two problems:

*Tsakane has R12. She spends R7. How much money does she have left?*

*Three friends must share 18 sweets equally. How many sweets must each friend get?*

The first is a simple addition problem, and the second a simple (sharing-type) division problem. There are, however, at least 20 different addition and subtraction problem types with different mathematical structures, and at least three division types. This is the case for all the basic operations. For example, the adult may classify the following problems as straightforward subtraction situations similar to the Tsakane problem given earlier, yet they have quite different structures:

*Tsakane has R5. She needs R12. How much money must she still get?*

*Tsakane has R12. Her brother has R5. How much more money does she have than her brother?*

We have found that some of the different mathematical structures identified as suitable and necessary for early word problems are definitely more difficult for learners, but that it also depends very much on the amount of exposure learners have had to a particular structure. If they have not met it before, it may be extremely difficult. Many teachers believe division to be difficult and consequently hold back the division-type problems until later and by this very act prevent learners from becoming familiar with the mathematical structures of the different division problems.
2. The number sizes and kinds of numbers involved

Number sense does not develop uniformly for all numbers simultaneously; it rather develops for ever-increasing number ranges. A child may for example feel comfortable with numbers below 40, but be quite lost with larger numbers. When the child is then confronted with a problem involving numbers beyond her number range, she cannot generalise the knowledge she already possesses about the properties of operations and numbers to include the larger numbers as well. For example, she may know that \(24 + 10 = 34\), but she does not generalise this to include \(78 + 10 = 88\). Even worse, these unfamiliar numbers create “blank spots” in the word problem that may destroy the structure of the problem for the child, making it inaccessible. How many adults are not confused by the statement: “Tim earns two thirds as much as Andy”? Replace the two thirds by a whole number, and the problem becomes very easy: “Tim earns three times as much as Andy.”

Compare the following two problems:

Peter and Lise work in the garden. Peter works for 1 hour and Lise works for 3 hours. Their father gives them R20. How must they share the money in a fair way?

Peter and Lise work in the garden. Peter works for 1½ (one and a half) hours and Lise works for 4½ (four and a half) hours. Their father gives them R20. How must they share the money in a fair way?

The solution to the first problem almost leaps to the eye, but the second problem requires careful thinking and calculation, yet the mathematical structures are identical and they even have the same solutions!

3. The context used for the problem

So-called real-life contexts are not necessarily accessible to learners. Learners experience “real life” very differently from adults, and are familiar with very different aspects of real life. Our “real life” problems are in any case not really real, but rather “pseudo-real”, though even fairly realistic contexts may create barriers to learners’ sense-making. Here are four of the many ways in which the context can act as a barrier to understanding.

- **Learners are not familiar with the context**
  
  Urban children may not know how a farm functions, how cows are milked, orchards planted, fertilizer applied. When I had to prepare materials for a project in the rural areas of a northern part of South Africa I found that accessible contexts were extremely limited – learners had no familiarity with radio or television timetables, no transport except taxis, no supermarkets, few foodstuffs and sweets, limited sports, and almost no exposure to magazines and newspapers.

- **The context has unpleasant connotations**
  
  This includes unpleasant racial, sexual and socio-economic overtones, but personal and/or family-based problems can also be serious handicaps. During one of my classroom visits the teacher posed a problem involving a family trip which I thought was very suitable for the group of children involved. One little girl withdrew completely; the teacher then remembered that the girl’s father had abandoned the family the previous week.

- **Limited contexts**
  
  When the mathematical activities presented to learners use only a limited set of contexts, the concepts and skills developed by learners may not be transferable to other contexts. A good example of this is the number skills that many street vendors have developed in the context of money matters which they cannot apply or find difficult to apply to similar problems in other contexts.

- **The problem has to be transformed or modelled by the learner before she can solve it**
  
  In mathematics, the problem itself is very seldom solved, rather it is modelled through introducing numbers, symbols, diagrams, etc. and these models are then manipulated to find the answer. When the following problem is posed to a young child: You have five sweets, and Dad gives you another two
sweets. How many sweets do you have now? she may write $5 + 2 = 7$. This is a model of the problem. At an earlier level she may draw 5 sweets, another 2 sweets, and count them all. The drawing is also a model. But at the level before that, the child cannot replace the problem with a model. If she does not have the sweets physically available, she cannot solve the problem. She cannot use counters or drawings to represent the sweets; she needs the sweets themselves. The teacher may therefore pose a problem involving counters or crayons that the learners may solve with ease, yet the very next day the same problem, now involving biscuits or rabbits, may be completely inaccessible to some learners.

4. The learners’ beliefs about what is expected of them

This includes beliefs about (i) the nature of mathematics and how mathematics is practised, (ii) how mathematics is learnt, and (iii) what the teacher values and how the teacher can be pleased. If the learner believes that mathematics is a ‘box of tricks’; that the algorithms (methods) to solve a problem have to be retrieved from memory and applied, and that the teacher expects the learner to solve the problem in a particular way using a method preferred by the teacher, the learner is almost forced into perceiving the problem posed not as something to make sense of, and to solve as such. The learner actually solves a different problem: that of figuring out what the teacher would like her to do. The problem, and careful reading of the problem, therefore becomes less important than guessing what the teacher’s expectations are (compare Schoenfeld, 1988, p. 85).

5. The teacher’s beliefs about the nature of mathematics and how mathematics is best learnt

This will determine where in the learning sequence word problems should feature and how they should be used (for example as starting points to stimulate thinking or purely as applications). It will also determine how the problems are selected and sequenced, and how the activities involving word problems are structured – are the problems discussed, will different voices be heard, will learners be encouraged to reflect by inviting explanations and argument? All the above, in turn, influence learners’ knowledge of the different problem structures and their willingness and ability to make sense of a given problem.

CONCLUSION

The word problems posed for early mathematical development do not use subject-specific terminology and are deliberately designed to offer situations that are accessible to young children, described in simple language. Yet many children have difficulty in understanding them. What can we learn from this?

I would like to suggest that functional literacy cannot be regarded as a general skill, but that it is domain-related, and deeply influenced by the socio-psychological factors that surround that particular domain for the individual reader or listener. The ability to read with understanding is not a “reading skill” as such, but depends as much on the reader’s previous knowledge, her ability to make sense of what she is doing, and her beliefs about what is expected of her, including what she herself believes to be the purpose of the reading act.

REFERENCES


Suggestions to writers

What is this journal for?

Learning and Teaching Mathematics is a journal of the Association for Mathematics Education of South Africa. This journal aims to provide a medium for stimulating and challenging ideas, offering innovation and practice in all aspects of mathematics teaching and learning. It seeks to inform, enlighten, stimulate, correct, entertain and encourage. Its emphasis is on addressing the challenges that arise in the learning and teaching of mathematics at all levels of education. It presents articles that describe or discuss mathematics teaching and learning from the perspective of a practitioner.

What type of submissions are we calling for?
The types of articles considered for publication in Learning and Teaching Mathematics are:

- **Ideas for teaching and learning**: articles in this section report on classroom activities and good ideas for teaching various mathematics topics. This includes worksheets, activities, investigations etc.

- **Letters to the editors**: discussion pieces that raise important issues on the teaching and learning of mathematics and current curriculum innovations. Views and news on current initiatives.

- **Kids say and do the darndest things**: personal anecdotes of something mathematical that has happened in a classroom.

- **Window on a Child’s Mind**: description of a classroom event that you want the Journal to respond to.

- **A day in the life of...** includes stories about a head of department, a maths teacher, an NGO worker etc.; it could also be an account of a visit to another mathematics classroom... another school... another country...

- **Reviews**: reviews of maths books, school mathematics textbooks, videos and movies, resources including apparatus and technology etc.

- **Webviews**: reviews of mathematics education related websites.

- **Help wanted** is a question and answer column: teachers can send their questions on teaching specific topics or aspects to this column for fellow colleagues in the AMESA community to respond to.

What are the technical requirements for the submission of articles?

Articles should not exceed 3 000 words and must be written in English. Articles as short as 300 words are also accepted and of course many of our categories such as “Question and Answers”, “Kids say and do the darndest things”, “letters to the editors” and so forth can be even shorter. Articles should include the title, author’s name, institution and full postal address, email and contact telephone numbers of the author.

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Post the completed application form (with the necessary fee) to: AMESA Membership, P.O. Box 54, WITS, 2050
Only if payment is by credit card (you must sign) or bank transfer may you e-mail or fax the form.
Enquiries: Tel: 011 484-8917  Fax: 011 484-2706 or 0865535042  E-mail: Membership@amesa.org.za  Valid for 2012