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LTM Cover Problem, June 2008

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The cover of the June 2008 issue of *Teaching and Learning Mathematics* depicted the following lovely, geometric result, though with no proof given anywhere in the journal: "A cyclic quadrilateral defines four arcs. The lines joining the midpoints of opposite arcs are perpendicular." An interactive sketch by the author is available online at: http://math.kennesaw.edu/~mdevilli/cyclic-quad-arcs.html

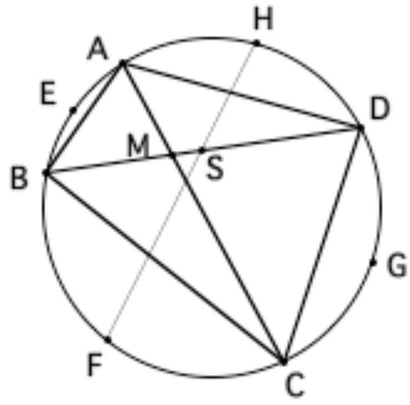


Figure 1

By chance, I happened to have produced a proof of this result as a lemma in the proof of another result on p. 190 of my *Some Adventures in Euclidean Geometry* book. Consider Figure 1 with *ABCD* a cyclic quadrilateral and *E*, *F*, *G* and *H* the midpoints of the arcs as indicated. We are required to prove that *HF* is perpendicular to *EG*.

This is certainly one of those results that can be quite challenging to prove until maybe in a moment of frustration, one constructs the diagonals of the cyclic quadrilateral. Then suddenly one might notice that the angle bisectors of the angles formed by the diagonals are respectively parallel to *HF* and *EG*. And then as the late Tickey de Jager would often say in his own enigmatic way "now it's plain sailing", well almost!

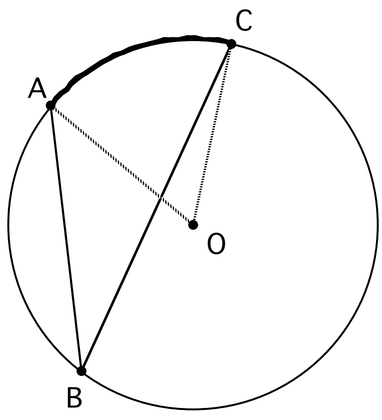


Figure 2

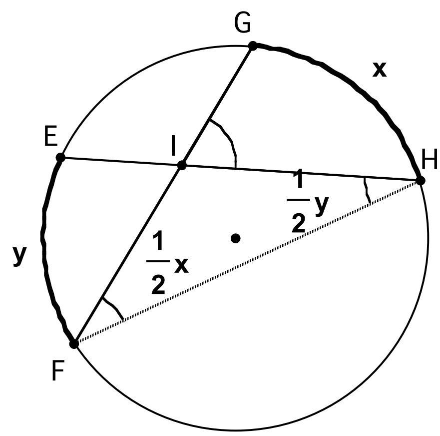


Figure 3

In any circle, the arc angle *AC* is defined as the angle subtended by it at the centre of the circle, e.g. $\angle\widehat{AC} = \angle AOC$. A familiar result is that the arc angle is twice the angle subtended by the arc on the circumference, i.e. $\angle ABC = \frac{1}{2}\left(\angle\widehat{AC}\right)$ (Figure 2). Now applying this to a configuration as shown in Figure 3, the result $\angle GIH = \frac{1}{2}\left(\angle\widehat{EF} + \angle\widehat{GH}\right)$, follows easily using the exterior angle theorem for a triangle. We can rephrase this result by saying the angle *GIH* is proportional to, or measured by, half the sum of the arcs *EF* and *GH*.

Now, in the original figure, the angle *AMD* is measured by half the sum of the arcs *AD* and *BC*. *HF* meets the diagonal *BD* in a point, *S*, inside the circle; hence angle *HSD* is measured by half the sum of the arcs *HD* and *BF*, and is therefore equal to half of the angle *AMD*, i.e. the line *FH* is parallel to the angle bisector of angle *AMD*. Similarly, *EG* is parallel to the angle bisector of angle *AMB*; but the angle bisectors are perpendicular to each other. Hence *FH* and *EG* are also perpendicular.

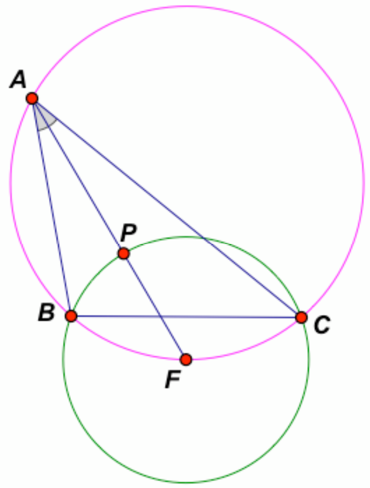
A relatively easy follow up problem might be: prove that the respective perpendiculars from the midpoints of the arcs to the opposite sides of the cyclic quadrilateral form another cyclic quad (when not concurrent). Intriguingly, the cyclic quadrilateral so formed is equi-angled to the original.

Reference

De Villiers, M. (2009). *Some Adventures in Euclidean Geometry*. Dynamic Mathematics Learning: Pinetown. (Available for purchase as downloadable PDF or printed book at http://www.lulu.com/product/download/some-adventures-in-euclidean-geometry/5414958)

Two additional challenges

1. Given any triangle *ABC*, construct the angle bisector of angle *BAC* and extend to meet the circumcircle of *ABC* in *F*. Prove that *F* is the midpoint of arc *BC*, and if a circle with *F* as centre and *FB* as radius is constructed, then the point *P*, the intersection of this new circle and *FA*, is the incentre of triangle *ABC*. (For an interactive sketch online go to: http://math.kennesaw.edu/~mdevilli/cyclic-incentre-result.html)



The LTM Cover problem, and the result above, can as shown in *Some Adventures in Euclidean Geometry*, pp. 189-191, be used to prove the following interesting result.

2. If the respective incentres, *P*, *Q*, *R* and *S* of triangles *ABC*, *BCD*, *CDA* and *DAB* of a cyclic quadrilateral *ABCD* are constructed, then *PQRS* is a rectangle. (For an interactive sketch online go to: http://math.kennesaw.edu/~mdevilli/cyclic-incentre-rectangle.html)